Determining the Position and Motion of a Vessel from Celestial Observations

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from Celestial Observations

George H. Kaplan
U.S. Naval Observatory

Abstract   Although many mathematical approaches to the celestial fix problem have been published, all of them fundamentally assume a stationary observer. Since this situation seldom occurs in practice, methods have been developed that effectively remove the observer's motion from the problem before a fix is determined. As an alternative, this paper presents a development of celestial navigation that incorporates a moving observer as part of its basic construction. This development allows recovery of the information on the vessel's course and speed contained in the observations. Thus, it provides the means for determining, from a suitable ensemble of celestial observations, the values of all four parameters describing a vessel's rhumb-line track across the Earth: latitude and longitude at a specific time, course, and speed. In many cases, this technique will result in better fixes than traditional methods.

Key Words   celestial navigation, celestial fix, motion of observer

Introduction

Celestial navigation is the oldest and most fundamental means of determining position at sea. Although now seldom employed as the primary means of navigation, it is still widely used. For example, despite the ubiquity of GPS receivers in modern U.S. defense systems, celestial navigation is practiced every day on ships of the U.S. Navy. Navy policy requires two independent means of position determination, and celestial navigation backs up GPS as an independent, absolute, world-wide, low-cost, self-contained system. Although the demise of celestial navigation has been predicted many times, reports of its death are, in the words of Mark Twain, greatly exaggerated.

The object of celestial navigation, as traditionally practiced, is the determination of the latitude and longitude of a vessel at a specific time, through the use of observations of the altitudes of celestial bodies. The chart-based approach, developed in the last century by Sumner and St.-Hilaire, translates each celestial altitude observation into a line of position (LOP) on the surface of the Earth. In principle, a series of observations defines a group of intersecting LOPs and this intersection represents the observer's position—the fix. Even before computers became small and cheap enough for routine shipboard use, specially designed navigational almanacs and precomputed sight-reduction tables made the procedure of finding position at sea relatively simple and free of laborious hand calculations. Chart-based celestial navigation employs some extremely clever schemes whereby a problem that is simple in concept, but tedious in practice, has been made relatively easy to solve. The method does involve some approximations (such as the use of straight-line LOPs), but given the accuracy of the observations made with hand-held sextants (of order $\pm 1'$), these errors are usually regarded as being insignificant.

The availability of inexpensive compact computing power over the last two decades motivated a re-examination of the celestial fix problem. For most authors, the objective was to develop robust mathematical solutions to the problem so that the algorithms could be readily converted into reliable computer or calculator programs. A number of innovative mathematical approaches to celestial navigation were formulated.

Two independent papers [1, 2] present an exact mathematical solution to a two-body fix. In [1], the development is based on the celestial coordinates of the two observed bodies, while in [2], the development is based on the geometry of the two LOPs. Both solutions are suitable for hand-held calculators. A less elegant iterative scheme for solving a two-body fix is given in [3].

When there are more than two observations, the problem is overdetermined and least-squares techniques are appropriate. A least-squares approach for a multi-star fix is presented in [4] based on the plane geometry and straight lines formed by LOPs near the estimated position. The method is a direct mathematical translation of chart-based navigation. It was developed independently at the Royal Greenwich Observatory

(RGO) and has been adopted by RGO in the publication *Compact Data for Navigation and Astronomy* [5]. The next issue of *Compact Data* will include a PC diskette with software that implements it. On page 282 of the *Nautical Almanac* [6], the section titled "Position from intercept and azimuth using a calculator" uses this algorithm.

Several other approaches to the multi-body problem have been published. A least-squares method based directly on the equation for a body's altitude as a function of latitude and longitude is developed in [7]. A vector-matrix approach to multi-body fixes has been published [8]. An alternative least-squares algorithm for sight reduction problems, using a Lagrange multiplier for normalization, is presented in [9], and applied to a vector-matrix formulation of the problem similar to that in [8]. The vector-matrix formulations, while elegant, have the disadvantage that the solution is obtained for a set of three parameters that have to be transformed to latitude and longitude. The transformations for this are straightforward and are given in the papers cited. However, the corresponding transformations of the statistical uncertainties (including the covariance needed to construct an error ellipse) are much less tidy and have not been published.

As varied as these algorithms are, the fundamental developments for all of them assume two or more co-located observations, which requires either a stationary observer or simultaneous sights. Normally, of course, the observations are taken sequentially from a moving ship. Therefore, something has to be done to account for the change in the observer's position during the time required to make the observations. Conventional schemes for doing so are reviewed in [10]; they effectively transform a moving-observer problem to a fixed-observer problem. For example, a commonly used procedure is the addition of motion-of-observer corrections to the observed altitudes. The most important weakness of such procedures is well known: they require data on the motion of the observer's ship over bottom, and the course and speed values used may not be accurate. The accuracy of these quantities is usually limited by our inexact knowledge of the local current. (The problems of running fixes are described in [11], pp. 136–137.) The errors involved are such that, for sights made with ordinary hand-held sextants, difficulties may arise for observations spread over more than about 0.5 h. If automated star trackers or similar high-accuracy devices were used for shipboard celestial navigation, observations spread over only a few minutes might be problematic.

Fortunately, the observations themselves contain information on the actual track of the vessel, so the possibility exists that the sight-reduction procedure can be made self-correcting. In principle, given enough observations, suitably distributed in time and azimuth, we should be able to obtain an estimate of the average over-bottom track of the vessel as part of the solution for the fix. The next two sections of this paper, together with the Appendix, present a development of celestial navigation that incorporates a moving observer as part of its basic construction. Such an approach correctly represents the propagation of positional error along the observer's track, here considered to be a rhumb line (loxodrome). Furthermore, the procedure allows us, under certain conditions, to recover information on the vessel's actual course and speed from the observations. This new algorithm includes the observer's motion as an essential part of the mathematics of celestial navigation, rather than as an add-on.

Celestial Navigation as an Orbit Correction Problem

Suppose we are given a series of observations taken over an extended period of time from a moving vessel. Is there a way to mathematically develop celestial navigation that includes the vessel's motion in the problem from the outset? Further, can such a development allow us to exploit the observations to correct our initial estimates of the course and speed of the vessel, as well as to provide a fix for a given time?

Our problem is quite similar to "orbit correction" problems faced by astronomers who deal with the dynamics of solar system bodies. Given a series of observations of some moving object in the solar system—an artificial satellite, a deep-space probe, an asteroid, or a planet—we want to be able to compute the position of the object at any given time. We know the laws of motion that the body obeys, but there are an infinite set of physically possible trajectories. Therefore, we must use the observations to determine the initial conditions, or orbital elements. The orbital elements are six parameters that specify the object's position and motion, in three dimensions, at a designated time. Once these six parameters have been determined, the object's position at any other time can be computed. The problem is the same regardless of what kind of observations are available. The observations may consist of simultaneous measurements of both celestial

coordinates, or the observations may be only of range (distance). In the latter case we use a series of one-dimensional observations to solve a six-dimensional problem. As long as we have at least six observations (suitably distributed) the problem is solvable.

The running fix problem in celestial navigation is analogous. The moving object of interest is the observer's ship. The fact that the observations are taken *from* the moving object rather than *of* the moving object does not change the nature of the problem. However, in celestial navigation, the problem is four-dimensional rather than six-dimensional because ships are constrained to move over the two-dimensional surface of the Earth. The sailing formulas for rhumb-line tracks are the "laws of motion." We have a series of one-dimensional observations—sextant altitudes—from which we wish to determine, for a given time, the four "orbital elements" of the vessel: latitude, longitude, course, and speed. Once these have been determined, the vessel's position at any other time can be computed. As long as we have four or more observations, well distributed in azimuth and time, the problem is solvable.

The orbit correction problem is usually dealt with through differential correction, utilizing least squares. This requires that we have some initial knowledge of the trajectory of the object of interest, which is almost always the case in both astronomy and navigation. This allows us to make a reasonably accurate estimate of the value of any observed quantity (e.g., declination or altitude) for any given time. The small difference between the observed value and the computed estimate can be accounted for by corrections to selected parameters in our *a priori* model of the object's motion. The problem is said to be linearized because mathematically we expand the formula for the observed quantity to first order in the parameters that we want to correct. That expansion becomes the conditional equation for a least-squares solution.

The solution to the problem is straightforward in theory, but the algebra can be messy. Suppose we have a sextant altitude, hs, measured at time t. We can obtain an observed altitude, Ho, by applying the usual corrections for index error, dip, and refraction to hs. We can also generate a computed altitude, Hc, which is based on our initial estimates of the vessel's location at the time of the observation. The formula for Hc as a function of the observer's location and the celestial coordinates of the observed body is well known. Each observation's altitude intercept, $a = \text{Ho} - \text{Hc}$, corresponds to what astronomers refer to as an O–C, an "observed minus computed" value. We assume that our initial knowledge of the vessel's track is reasonably good and that the observed body's celestial coordinates are exactly known, so that $\text{Hc} \approx \text{Ho}$ and a is small. If the observation were perfect, a non-zero value for a would represent an error in Hc due to an incorrect estimated position. Expanding Hc in terms of small increments of the instantaneous latitude, ϕ, and longitude, λ:

$$a = \Delta\text{Hc} = \frac{\partial \text{Hc}}{\partial \phi}\Delta\phi + \frac{\partial \text{Hc}}{\partial \lambda}\Delta\lambda \tag{1}$$

Equation (1) can be readily evaluated for each observation, an approach taken in [7]. In the relation above, $\Delta\phi$ and $\Delta\lambda$ are corrections to the coordinates of the vessel at the observation time, t. However, what we really want are corrections to the latitude, longitude, course, and speed of the vessel at some reference time, t_0. Obviously, having correct values for these four parameters would allow us to compute the instantaneous latitude and longitude of the vessel at any other time. Symbolically,

$$\begin{aligned}
\phi &= f(\phi_0, \lambda_0, C, S, t - t_0) \\
\lambda &= g(\phi_0, \lambda_0, C, S, t - t_0)
\end{aligned} \tag{2}$$

where ϕ_0, λ_0, C, and S are, respectively, the vessel's latitude, longitude, course, and speed at time t_0. We do not place subscripts on C or S because we assume the vessel is sailing a rhumb-line track at a fixed speed so these parameters have constant values. The functions f and g represent the sailing formulas for latitude and longitude, respectively, for rhumb-line tracks (see [12]). (There is no reason this development could not also be made to apply to great-circle tracks.) Assume that initially we have estimates for the values of ϕ_0, λ_0, C, and S, but that these are in need of correction from the observations. That is, t_0, our reference time, is also the time for which we want a fix. Therefore, we want to rewrite equation (1) as a function of these four quantities. To do so we need to express $\Delta\phi$ and $\Delta\lambda$, which are themselves small quantities, in terms of

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small increments in the four parameters of interest:

$$\Delta\phi = \frac{\partial f}{\partial \phi_0}\Delta\phi_0 + \frac{\partial f}{\partial \lambda_0}\Delta\lambda_0 + \frac{\partial f}{\partial C}\Delta C + \frac{\partial f}{\partial S}\Delta S$$

$$\Delta\lambda = \frac{\partial g}{\partial \phi_0}\Delta\phi_0 + \frac{\partial g}{\partial \lambda_0}\Delta\lambda_0 + \frac{\partial g}{\partial C}\Delta C + \frac{\partial g}{\partial S}\Delta S$$

(3)

Substituting equations (3) into equation (1) and rearranging terms we obtain

$$a = \left(\frac{\partial \mathrm{Hc}}{\partial \phi}\frac{\partial f}{\partial \phi_0} + \frac{\partial \mathrm{Hc}}{\partial \lambda}\frac{\partial g}{\partial \phi_0}\right)\Delta\phi_0 + \left(\frac{\partial \mathrm{Hc}}{\partial \phi}\frac{\partial f}{\partial \lambda_0} + \frac{\partial \mathrm{Hc}}{\partial \lambda}\frac{\partial g}{\partial \lambda_0}\right)\Delta\lambda_0$$

$$+ \left(\frac{\partial \mathrm{Hc}}{\partial \phi}\frac{\partial f}{\partial C} + \frac{\partial \mathrm{Hc}}{\partial \lambda}\frac{\partial g}{\partial C}\right)\Delta C + \left(\frac{\partial \mathrm{Hc}}{\partial \phi}\frac{\partial f}{\partial S} + \frac{\partial \mathrm{Hc}}{\partial \lambda}\frac{\partial g}{\partial S}\right)\Delta S$$

(4)

Equation (4) expresses the altitude intercept in terms of small increments (errors) in the estimated values of ϕ_0, λ_0, C, and S, denoted $\Delta\phi_0$, $\Delta\lambda_0$, ΔC, and ΔS, respectively. These latter four quantities are unknowns to be determined, and once their values are known, we can correct our estimate of the vessel's track. The quantities within parentheses in equation (4) are the coefficients of the unknowns, which, along with the altitude intercept a, can be numerically evaluated for each observation. We originally postulated a perfect observation, and evaluating equation (4) for four such observations would allow a direct solution for the four unknowns. Of course, we never have perfect observations, so we obtain more than four and use the above equation as the conditional equation in a least-squares solution for the unknowns.

Equation (4) can also be expressed in matrix form:

$$a = \left(\begin{array}{cc} \frac{\partial \mathrm{Hc}}{\partial \phi} & \frac{\partial \mathrm{Hc}}{\partial \lambda} \end{array}\right) \left(\begin{array}{cccc} \frac{\partial f}{\partial \phi_0} & \frac{\partial f}{\partial \lambda_0} & \frac{\partial f}{\partial C} & \frac{\partial f}{\partial S} \\ \frac{\partial g}{\partial \phi_0} & \frac{\partial g}{\partial \lambda_0} & \frac{\partial g}{\partial C} & \frac{\partial g}{\partial S} \end{array}\right) \left(\begin{array}{c} \Delta\phi_0 \\ \Delta\lambda_0 \\ \Delta C \\ \Delta S \end{array}\right)$$

(5)

To numerically evaluate equation (4) for each observation, we need to expand the partial derivatives in equation (4) into algebraic expressions. There are ten partials required:

$$\frac{\partial \mathrm{Hc}}{\partial \phi}, \frac{\partial \mathrm{Hc}}{\partial \lambda}, \frac{\partial f}{\partial \phi_0}, \frac{\partial f}{\partial \lambda_0}, \frac{\partial f}{\partial C}, \frac{\partial f}{\partial S}, \frac{\partial g}{\partial \phi_0}, \frac{\partial g}{\partial \lambda_0}, \frac{\partial g}{\partial C}, \frac{\partial g}{\partial S}$$

These are obtained from three formulas: the equation for Hc as a function of ϕ and λ, and the sailing formulas represented by the functions f and g. In the Appendix, the expressions for the partials needed in equation (4) are evaluated.

Note that we can obtain a traditional fix by simply not solving for ΔC and ΔS, that is, by ignoring the second line of equation (4). That would be advisable if there were only a few observations, or the observations were poorly distributed in azimuth or tightly grouped in time.

Least-squares methods are based on the assumption that the errors of observation have a Gaussian distribution. Ideally, a non-zero value for $a = \mathrm{Ho} - \mathrm{Hc}$ arises from the incorrect values of the parameters used to compute Hc and from measurement noise in Ho. However, there are several other contributors to the value of a that can affect the solution. First, Ho is not a direct measurement, but is derived from the sextant altitude hs by applying several corrections. The most uncertain of these is atmospheric refraction, and small systematic errors (of order 0.1 arcmin) cannot be ruled out under some conditions. Second, a ship does not follow an exact rhumb-line course at constant speed, but is subject to random variations in wind, current, and steering. As illustrated in [11], a vessel's path over bottom is a somewhat irregular line. These deviations from a rhumb-line track are, of course, unknown and cannot be included in the calculation of Hc. The effect of these variations on the values of a is random if the typical time between observations is greater than the time scale of the variations themselves. (To make a more definitive statement, a quantitative model of the

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wind, current, and steering variations would be needed.) Given a sufficient number of such observations, then, the algorithm described in this paper yields the parameters for a kind of average rhumb-line track over bottom, which is, presumably, what is desired. More problematic are systematic changes in the current or wind that occur over time scales of hours. In such circumstances the ship's track may not be well represented by a single constant-speed rhumb line, yet the algorithm described in this paper will force fit the data to such a model. If the ship's track can be modelled as a series of connected rhumb lines, then a generalization of the algorithm described here can be used; see [13].

Procedure

An outline of the recommended computational procedure follows. Equations (A1) through (A9), referred to below, are given in the Appendix.

(1) Determine the time, t_0, for which the fix is to be computed. Usually t_0 will be within the time span represented by the observations. Estimate the values of the latitude ϕ_0, longitude λ_0, course C, and speed S for time t_0. The course and speed should include the set and drift of the current, that is, they should refer to the vessel's motion over bottom. If necessary, use the sailing equations (A7) (or equations (A5) for east-west tracks) to propagate the latitude and longitude forward from an earlier fix.

(2) Decide whether course and speed corrections are to be determined. These corrections should not be attempted unless there are at least eight observations with a good distribution of azimuths. Furthermore, for the determination to be reliable, the observations must span a significant period of time. What constitutes a significant period of time depends on the uncertainties in the vessel's course and speed over bottom and the accuracy of the observations. For hand-held sextant observations, the required period of time will generally be several hours or more, and twilight observations would have to be combined with day or night observations (see next two sections).

(3) Correct each observation for index error, dip, refraction, and other effects in the usual manner, converting each sextant altitude, hs, to an observed altitude, Ho.

(4) Take each of the observations in turn and perform the following operations on it:

(a) Use the sailing formulas to compute the estimated position (ϕ, λ) of the vessel at the time t of the observation. For east-west courses, use equations (A5), then set $\phi = \phi'$ and $\lambda = \lambda'$. For other courses, first compute ϕ' from equations (A5), then use equations (A7) to obtain ϕ and λ.

(b) From the appropriate ephemeris (almanac routine), determine the celestial coordinates (GHA, Dec) of the body observed at time t. Compute Hc using equation (A1), then compute the altitude intercept $a = \text{Ho} - \text{Hc}$.

(c) Compute the partial derivatives of the sailing formulas. For east-west courses, use equations (A6). For other courses, compute $\frac{\partial \phi'}{\partial \phi_0}$, $\frac{\partial \phi'}{\partial C}$, and $\frac{\partial \phi'}{\partial S}$ from equations (A6), compute $\frac{\partial \lambda}{\partial \phi}$ from equation (A8), then use equations (A9).

(d) Evaluate equation (4), the conditional equation for the observation. First compute $\frac{\partial \text{Hc}}{\partial \phi}$ and $\frac{\partial \text{Hc}}{\partial \lambda}$ from equations (A2). Then combine these with the partial derivatives previously computed to evaluate the quantities in parentheses in equation (4), i.e., the four coefficients of the unknowns. If corrections to course and speed are not to be determined, only the two coefficients in the first line of equation (4) need to be evaluated.

(e) Include the conditional equation for the observation in the least-squares solution. This may involve a routine that accepts as input the altitude intercept and the coefficients of the unknowns for the current observation. The coefficients for the current observation become a new row in what is called the *design matrix* of the least-squares solution. (For a description of least-squares techniques, see any text on scientific data analysis, for example, [14].)

(5) Compute the least-squares solution, based on all the observations. In the most general case, the solution will yield values of $\Delta \phi_0$, $\Delta \lambda_0$, ΔC, and ΔS. If corrections to course and speed are not being

determined, however, we will obtain only $\Delta\phi_0$ and $\Delta\lambda_0$ from the solution and must consider ΔC and ΔS to be zero. The corrections $\Delta\phi_0$, $\Delta\lambda_0$, and ΔC obtained from the solution will be expressed in radians, and ΔS will be expressed in km/h. Least-squares procedures also provide the uncertainties in the computed values of the unknowns as elements of a variance-covariance matrix. If needed, an error ellipse in the latitude-longitude plane can be constructed using the variances of latitude and longitude together with the latitude-longitude covariance (see, for example, [14]).

(6) Using the results of the least-squares solution, correct the initial estimates of ϕ_0, λ_0, C, and S:

$$\phi_0 \rightarrow \phi_0 + \Delta\phi_0$$
$$\lambda_0 \rightarrow \lambda_0 + \Delta\lambda_0$$
$$C \rightarrow C + \Delta C$$
$$S \rightarrow S + \Delta S$$

(7) Decide whether another iteration of the procedure is necessary. Although good results are often obtained in a single pass, a second is recommended to ensure that the results have converged. Generally, more than two iterations are needed only if the initial estimates of ϕ_0, λ_0, C, and S used in step (1) were badly in error. A general iteration decision can be based on the values of $\Delta\phi_0$, $\Delta\lambda_0$, ΔC, and ΔS; if they are all smaller than pre-established criteria, or significantly smaller than their own uncertainties, then another iteration is unnecessary. If another iteration is required, take the new values of ϕ_0, λ_0, C, and S, and redo the procedure above beginning at step (4).

The new values of ϕ_0 and λ_0 define the fix for time t_0. These, combined with the new values of C and S, define an estimate of the vessel's average rhumb-line track, over bottom, during the time the observations were taken.

Sample Calculation

A numerical example using this algorithm is illustrative. In the sample calculation below, a set of eight artificial sextant observations (hs values), spanning eight hours, was reduced using the above procedure, resulting in a position fix and course and speed estimates. The artificial observations were generated by a computer program that numerically integrated a hypothetical ship's rhumb-line track based on the data in Table 1 (north latitudes and east longitudes are positive).

<div align="center">

Table 1—Ship's Track Used in Sample Calculation

Date:	10 Apr 1995
Time:	14:00:00 UTC
Latitude:	$+45.0000°$
Longitude:	$-50.0000°$
Course:	$330.00°$
Speed:	20.00 kn

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The data in Table 1, and any positions derived from them, constitute "truth" for this case. Astronomical positions of the celestial bodies chosen were computed that are consistent with those in the 1995 *Nautical Almanac* and transformed to local altitudes. Atmospheric refraction was included but the height of eye was considered to be zero, and the time scale UT1 was assumed to be identical to UTC (that is, DUT1= UT1–UTC=0). Random errors with a 0.7 arcmin standard deviation were added to the calculated hs values. The artificial observations, all for 10 April 1995 (UTC), are given in Table 2.

<div align="center">

Table 2—Artificial Observations for Sample Calculation

UTC	Object	hs (°)	UTC	Object	hs (°)
15:19:22.4	Sun (L)	52.2903	22:30:15.6	Mars	59.3888
18:45:05.9	Moon (U)	19.1563	22:46:27.5	Kochab	44.0681
20:01:13.8	Sun (L)	20.2358	23:02:39.4	Capella	54.3544
21:01:10.0	Moon (U)	39.4857	23:20:28.5	Rigel	14.3925

</div>

The data in the left column of Table 2 represent daytime observations, and those in the right column represent evening twilight observations (sunset was at 22:09). The Sun data apply to the lower limb (L) and the Moon data apply to the upper limb (U), as would be appropriate for observations from the hypothetical ship at the times listed. The Moon's phase was 0.75 (age 11 days).

Once computed and stored in a file, these artificial observations were used as input to a separate computer program that implemented the new algorithm described in this paper. This second program was given the incorrect data on the ship's position and motion listed in Table 3.

Table 3—Incorrect Data on Ship's Track Supplied to New Algorithm

Date:	10 Apr 1995
Time:	14:00:00 UTC
Latitude:	$+45.4000°$
Longitude:	$-49.5000°$
Course:	$332.00°$
Speed:	18.50 kn

The test of the algorithm is whether, given only the eight observations (Table 2), it can determine the true track of the vessel (Table 1), limited only by the error in the observations. The date and time for the desired fix, t_0, were chosen to be 10 Apr 1995 at 23:00:00. Latitude, longitude, course, and speed were to be determined. Based on the incorrect data on the ship's track it was given, the second program computed the estimated position for time t_0 to be $\phi_0 = +47.8493°$, $\lambda_0 = -51.3907°$. It also computed the data in Table 4 for the eight observations.

Table 4—Estimates of Sight-Reduction Quantities for Observations

UTC	Est ϕ	Est λ	Object	Ho	Hc	a	Zn
	°	°		°	°	′	°
15:19:22.4	45.7601	−49.7726	Sun (L)	52.2774	51.9007	+22.61	179.5
18:45:05.9	46.6932	−50.4876	Moon (U)	19.1091	19.5297	−25.24	96.8
20:01:13.8	47.0386	−50.7553	Sun (L)	20.1913	19.6929	+29.90	260.1
21:01:10.0	47.3104	−50.9672	Moon (U)	39.4656	39.6996	−14.04	126.2
22:30:15.6	47.7144	−51.2843	Mars	59.3790	59.3585	+ 1.23	154.9
22:46:27.5	47.7879	−51.3422	Kochab	44.0510	44.4809	−25.80	22.4
23:02:39.4	47.8613	−51.4002	Capella	54.3425	53.9105	+25.92	287.7
23:20:28.5	47.9421	−51.4641	Rigel	14.3293	13.7379	+35.48	240.6

In Table 4, the altitude intercept, a, is $Ho - Hc$ and positive values indicate "toward." The Ho and Hc values are topocentric and, for the Sun and Moon, they refer to the limb observed. The computed azimuth values, Zn, are not used by the algorithm and are listed in the table only for reference.

As it processed each observation, the program computed the values of the coefficients of the four unknowns in equation (4). It sent these coefficient values, along with the value of a, to the least-squares routine, where a new row in the design matrix was constructed. All the observations were given equal weight. After all eight observations had been processed in this way, the least-squares solution was formed. The result, for time $t_0 = 23:00:00$ UTC, is given in Table 5.

Table 5—Least-Squares Solution—Iteration 1

Parameter Values		Parameter Correlations			
$\Delta\phi_0$ -0.249370 ± 0.008981 °		$\Delta\phi_0$	$+1.000$	-0.038	$+0.108$	$+0.526$
$\Delta\lambda_0$ -0.787800 ± 0.012634 °		$\Delta\lambda_0$	-0.038	$+1.000$	$+0.669$	-0.492
ΔC -2.175234 ± 0.579230 °		ΔC	$+0.108$	$+0.669$	$+1.000$	-0.397
ΔS $+1.505770 \pm 0.150014$ kn		ΔS	$+0.526$	-0.492	-0.397	$+1.000$
			$\Delta\phi_0$	$\Delta\lambda_0$	ΔC	ΔS

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In Table 5, the values and errors of the unknowns (1σ mean errors) have been converted to degrees and knots from radians and kilometers per hour, respectively. The parameter correlations were derived directly from the variance-covariance matrix from the solution and indicate the degree to which pairs of parameters were statistically coupled in the solution. The solution was well behaved and the mean error of unit weight was computed to be 0.8 arcmin. The program then added the parameter values from Table 5 to the estimated (incorrect) data on the ship's track it had been using. The results for 23:00:00 UTC are summarized in Table 6.

Table 6—Results of Solution for Ship's Track—Iteration 1

	Truth	Estimate	Solution	Truth–Estimate	Truth–Solution
ϕ_0	+47.5972	+47.8493	+47.5999	−0.2521	−0.0027
λ_0	−52.1640	−51.3907	−52.1785	−0.7733	+0.0145
C	330.00	332.00	329.82	−2.00	+0.18
S	20.00	18.50	20.01	+1.50	−0.01

The units used in Table 6 are the same as in previous tables. Obviously a great improvement in the data on the ship's track was made. Of course, in a real situation, "Truth" would not be known; the question at this point would be whether another iteration of the procedure would be advisable, using the updated values of ϕ_0, λ_0, C, and S now available. We note that in Table 5, the absolute values of all the unknowns are greater than their errors, a condition that suggests that another iteration might be fruitful. Another pass through the procedure was therefore made, using the values listed under the "Solution" column in the Table 6 as the new data on the ship's track. For the second iteration the program obtained the data given in Tables 7 and 8.

Table 7—Least-Squares Solution—Iteration 2

Parameter Values		Parameter Correlations			
$\Delta\phi_0$	-0.004465 ± 0.009198 °	$\Delta\phi_0$	+1.000 −0.034 +0.131 +0.509		
$\Delta\lambda_0$	$+0.004870 \pm 0.012817$ °	$\Delta\lambda_0$	−0.034 +1.000 +0.662 −0.510		
ΔC	$+0.076011 \pm 0.543305$ °	ΔC	+0.131 +0.662 +1.000 −0.402		
ΔS	$+0.021225 \pm 0.154628$ kn	ΔS	+0.509 −0.510 −0.402 +1.000		
			$\Delta\phi_0 \quad \Delta\lambda_0 \quad \Delta C \quad \Delta S$		

Table 8—Results of Solution for Ship's Track—Iteration 2

	Truth	Estimate	Solution	Truth–Estimate	Truth–Solution
ϕ_0	+47.5972	+47.5999	+47.5954	−0.0027	+0.0018
λ_0	−52.1640	−52.1785	−52.1737	+0.0145	+0.0097
C	330.00	329.82	329.90	+0.18	+0.10
S	20.00	20.01	20.03	−0.01	−0.03

Table 8 indicates that the second iteration was an improvement over the first iteration. Note that the "Solution" from the first iteration appears as the "Estimate" for the second iteration.

The new algorithm did recover the ship's true track quite well. For this case, the error ellipse in the latitude-longitude plane is nearly a circle, with a circular probable error (CEP) of 0.63 nmi. The ship's true position for 23:00 is well within the circle; the actual error of the fix is 0.41 nmi.

Other numerical experiments were conducted to provide a broader assessment of the algorithm. The first was to repeat the above calculations without solving for course and speed. The fix position obtained (for 23:00) was $\phi_0 = +47.5471°$, $\lambda_0 = -52.1115°$, an error of 3.67 nmi. Clearly, in this case, the entire set of observations should not be used for a fix *without* solving for course and speed. However, a more common practice would be to use just the twilight observations. When the four twilight observations were used to obtain a fix without solving for course and speed, the resulting position was $\phi_0 = +47.5867°$, $\lambda_0 = -52.1752°$, an error of 0.78 nmi. So the new algorithm, used with the entire set of eight observations, improved the

23:00 fix by about a factor of two. The algorithm becomes progressively more advantageous as the accuracy of the observations increase, as Table 9, computed for the case being considered, shows.

Table 9—Algorithm Comparison

Random Error of Observations	— New Algorithm —		— Standard Fix —	
	CEP	Actual Error	CEP	Actual Error
′	nmi	nmi	nmi	nmi
0.7	0.63	0.41	0.80	0.78
0.3	0.27	0.17	0.51	0.53
0.1	0.090	0.058	0.36	0.45
0.02	0.018	0.013	0.31	0.42
0	0.001	0.001	0.29	0.42

In Table 9, "New Algorithm" refers to a solution for ϕ_0, λ_0, C, and S from all eight observations, while "Standard Fix" refers to a solution for ϕ_0 and λ_0 from the four twilight observations. (If the standard fix had included all eight observations, its results would have been considerably worse.) Two iterations were performed in all cases. As the errors in the observations decrease, the two methods separately converge, but to different answers—one correct, the other not. For the standard fix, neither the CEP nor the actual error decreases in proportion to the random error of the observations because of the underlying systematic error from the incorrect course and speed. Geometrically, the incorrect course and speed prevent the advanced LOPs from converging toward a single point. Table 9 neatly summarizes the advantages of the new algorithm for accurate observations. Of course, we have been assuming that the ship's track is well modelled by a constant-speed rhumb line over the eight hours that the observations span. If that is not the case, the new algorithm will be less successful.

Discussion

The algorithm described in this paper has been incorporated into a new software package for celestial navigation. The package was developed by the U.S. Naval Observatory in response to a Navy requirement for a modern computer implementation of various celestial navigation procedures. Although now implemented only on PCs, this software will eventually become a module in the Navy's Navigation Sensor System Interface (NAVSSI) [15].

The algorithm has been extensively tested using artificially generated data, both with and without random errors. Many examples found in navigation texts have been reduced again using it, and the results compared with other sight-reduction algorithms. Additionally, because the software that implements the algorithm has been beta tested on board deployed Navy vessels, the method has been checked with real data. The algorithm works well and is robust. Often, adequate answers are obtained in a single iteration, although two iterations are the norm. Covariances among the parameters being solved for are usually low with reasonable observational data. The tests with perfect artificial observations have demonstrated the mathematical correctness of the algorithm.

However, the full power of the algorithm has probably not been used so far in practice. As previously noted, the most common navigation procedure is a round of sights, in which a small number of observations (usually three to five) are taken within a short period of time, in twilight, and reduced to determine a fix. The accuracies of hand-held sextant observations from a moving ship are typically of order one arcminute and the accuracy of the resulting fix may range from a half to several nautical miles. For these reductions almost any sight-reduction procedure is adequate and the algorithm described in this paper does not have significant practical advantages over others. Course and speed corrections cannot be determined with such a limited observation set (and the software that implements the new algorithm will not attempt to do so). For such cases, all reasonable algorithms give essentially the same answer.

The advantages of the algorithm presented in this paper emerge when navigational practice is extended beyond the usual twilight round of sights or noon Sun line. It would be especially suited for an automated star tracker with an artificial horizon that could observe all night (or, in the near infrared, during the day).

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Star trackers in various forms are currently used on aircraft, missiles, and spacecraft, and accuracies of a few arcseconds are attainable. But even with ordinary hand-held marine sextants, more flexibility in navigational procedures is possible than is usually practiced. For example, at high latitudes, long periods of twilight allow for extended sets of observations. Sun-Moon fixes are geometrically possible during about half of all days. (Navigators have avoided using the Moon as a target because of the extra corrections that Moon observations require and the false horizons that can appear under the Moon at night. Sight reduction software such as that described in this paper eliminates the first problem, and the second does not arise during the day.) Sun or Moon observations from early or late in the day can be combined with twilight observations. Observations of the stars and planets are possible at night near full Moon when the sky is bright enough to make the horizon visible. It is these cases—involving an extended sequence of observations, very accurate observations, or both—that the algorithm described in this paper was designed for. Furthermore, the algorithm can be generalized to observation sets taken over multiple voyage legs [13].

Conclusion

A variety of algorithms for celestial navigation have been published, but all are fundamentally based on the concept of a fixed observer. For these algorithms, the motion of the observer is a nuisance external to the basic development. Either explicitly or implicitly, the moving-observer case must be transformed to a fixed-observer case, a process that requires data on the observer's course and speed. Errors in the course and speed values used can propagate into the celestial fix.

As an alternative, this paper presents an algorithm for celestial navigation based upon the concept of a moving observer. This algorithm can use the information contained in a set of celestial observations on the track of the observer's ship across the surface of the Earth. For a suitable set of observations, the algorithm allows for the determination of the course and speed of the observer's vessel, as well as its position at a given time. Fully exploiting this algorithm would require some changes in navigational practice or equipment. However, the algorithm also yields a correct fix from a traditional round of sights. It has been extensively tested. The algorithm has been incorporated into a new software package that will soon be a standard navigation tool on ships of the U.S. Navy.

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Appendix

Expressions for Terms in the Conditional Equation

To use equation (4) as a conditional equation for a least-squares solution for various unknowns, we need to be able to numerically evaluate the quantities within parentheses in the equation. These quantities are the coefficients of the unknowns. For each observation, the following ten partial derivatives must be computed:

$$\frac{\partial \text{Hc}}{\partial \phi}, \ \frac{\partial \text{Hc}}{\partial \lambda}, \ \frac{\partial f}{\partial \phi_0}, \ \frac{\partial f}{\partial \lambda_0}, \ \frac{\partial f}{\partial C}, \ \frac{\partial f}{\partial S}, \ \frac{\partial g}{\partial \phi_0}, \ \frac{\partial g}{\partial \lambda_0}, \ \frac{\partial g}{\partial C}, \ \frac{\partial g}{\partial S}$$

The first two will be derived from the usual spherical-trig equation for Hc, the next four are obtained from the sailing formula for latitude (the function f), while the last four are obtained from the sailing formula for longitude (the function g). The equation for Hc is elementary:

$$\text{Hc} = \arcsin\left[\sin \text{Dec} \sin \phi + \cos \text{Dec} \cos \phi \cos(\text{GHA} + \lambda)\right] \tag{A1}$$

from which we immediately obtain

$$\frac{\partial \text{Hc}}{\partial \phi} = \sec \text{Hc} \left[\sin \text{Dec} \cos \phi - \cos \text{Dec} \sin \phi \cos(\text{GHA} + \lambda)\right]$$

$$\frac{\partial \text{Hc}}{\partial \lambda} = -\sec \text{Hc} \left[\cos \text{Dec} \cos \phi \sin(\text{GHA} + \lambda)\right] \tag{A2}$$

For rhumb-line tracks, the sailing formulas for latitude and longitude, including terms for the oblateness of the Earth, are given in [12]. The sailing formulas contain the quantities M and N, which represent the radii of curvature of the Earth's surface in the meridian and the prime vertical, respectively, in kilometers. (The prime vertical is the plane containing the normal to the Earth's surface at the location of interest, perpendicular to the local meridian.) The curvatures M and N at a given latitude, ϕ, are computed from the Earth's equatorial radius a, in kilometers, and the dimensionless eccentricity of the Earth's ellipsoid, e:

$$M = \frac{a(1 - e^2)}{(1 - e^2 \sin^2 \phi)^{3/2}}$$

$$N = \frac{a}{(1 - e^2 \sin^2 \phi)^{1/2}} \tag{A3}$$

Here, e is related to the Earth's flattening, f, by $e^2 = 2\text{f} - \text{f}^2$ (e is sometimes referred to in geodesy texts as the "first eccentricity"). The WGS-84 values are $a = 6378.137$ km and f $= 1/298.257223563$. In this appendix we will use metric units for distance and speed in order to avoid confusion between nautical miles and arcminutes—on an oblate Earth, nautical miles and arcminutes do not, in general, correspond. One International Nautical Mile is exactly 1.852 km, so 1 kn = 1.852 km/h. Angular quantities will be expressed in radians.

The two curvatures M and N occur in the sailing formulas, and we will be needing the partial derivatives of M, M^{-1}, and N^{-1} with respect to ϕ:

$$\frac{\partial M}{\partial \phi} = \frac{3ae^2(1 - e^2)\sin 2\phi}{2(1 - e^2 \sin^2 \phi)^{5/2}}$$

$$\frac{\partial M^{-1}}{\partial \phi} = -\frac{3e^2 \sin 2\phi \, (1 - e^2 \sin^2 \phi)^{1/2}}{2a(1 - e^2)} \tag{A4}$$

$$\frac{\partial N^{-1}}{\partial \phi} = -\frac{e^2 \sin 2\phi}{2a(1 - e^2 \sin^2 \phi)^{1/2}}$$

To evaluate the partial derivatives we need for equation (4), we will start with the simplified sailing formulas given in [12]. These formulas are approximations, but they are accurate to within about 10 m over tracks where the excursion in latitude does not exceed about 15 km ($\approx$10 nmi). They are much less accurate over larger latitude ranges. Thus, these formulas could be used to accurately project the motion of the vessel over one-half to one hour of sailing, or longer periods if the course is east-west or very nearly so. These formulas, which give the latitude and longitude of the vessel as a function of the time, t, are:

$$\phi' = \phi_0 + \frac{S\,(t-t_0)\cos C}{M_0}$$

$$\lambda' = \lambda_0 + \frac{S\,(t-t_0)\sin C}{N_0 \cos \phi_{avg}}$$

where $\qquad \phi_{avg} = \tfrac{1}{2}(\phi_0 + \phi') \qquad\qquad$ (A5)

$$M_0 = M(\phi_0) = \frac{a(1-e^2)}{(1-e^2\sin^2\phi_0)^{3/2}}$$

$$N_0 = N(\phi_0) = \frac{a}{(1-e^2\sin^2\phi_0)^{1/2}}$$

Here, ϕ_0 and λ_0 are the latitude and longitude of the reference (initial or known) position and t_0 is the time at which the vessel crosses that position. The quantities M_0 and N_0 are the values of the curvatures M and N at the at the reference latitude ϕ_0. The vessel's speed, S, is expressed in km/h = kn $\times$ 1.852, and the time difference, $t-t_0$, is expressed in hours. The primes on ϕ' and λ' are simply meant to indicate that these are approximations that are accurate only over certain (short or east-west) tracks. More generally, these formulas can be used to provide a first approximation to the latitude and longitude at time t. The above sailing formulas are modifications of the usual "middle latitude" sailing formulas given on pp. 364–365 of Bowditch [11].

We can identify the expressions for ϕ' and λ' given above with the functions f and g referred to in equations (2), (3), and (4). That puts us in a position to use equations (A5) to derive the partial derivatives needed for equation (4):

$$\frac{\partial f}{\partial \phi_0} = \frac{\partial \phi'}{\partial \phi_0} = 1 + S\,(t-t_0)\,\cos C\,\frac{\partial M_0^{-1}}{\partial \phi_0}$$

$$\frac{\partial f}{\partial \lambda_0} = \frac{\partial \phi'}{\partial \lambda_0} = 0$$

$$\frac{\partial f}{\partial C} = \frac{\partial \phi'}{\partial C} = -\frac{S\,(t-t_0)\,\sin C}{M_0}$$

$$\frac{\partial f}{\partial S} = \frac{\partial \phi'}{\partial S} = \frac{(t-t_0)\,\cos C}{M_0}$$

$$\frac{\partial g}{\partial \phi_0} = \frac{\partial \lambda'}{\partial \phi_0} = \frac{S\,(t-t_0)\,\sin C}{\cos \phi_{avg}}\left[\frac{\partial N_0^{-1}}{\partial \phi_0} + \frac{\tan \phi_{avg}}{2N_0}\left(1 + \frac{\partial \phi'}{\partial \phi_0}\right)\right] \qquad\qquad \text{(A6)}$$

$$\frac{\partial g}{\partial \lambda_0} = \frac{\partial \lambda'}{\partial \lambda_0} = 1$$

$$\frac{\partial g}{\partial C} = \frac{\partial \lambda'}{\partial C} = \frac{S\,(t-t_0)}{N_0 \cos \phi_{avg}}\left[\cos C + \sin C\,\tan \phi_{avg}\left(\frac{1}{2}\frac{\partial \phi'}{\partial C}\right)\right]$$

$$\frac{\partial g}{\partial S} = \frac{\partial \lambda'}{\partial S} = \frac{(t-t_0)\,\sin C}{N_0 \cos \phi_{avg}}\left[1 + S\,\tan \phi_{avg}\left(\frac{1}{2}\frac{\partial \phi'}{\partial S}\right)\right]$$

In the above expressions, $\frac{\partial M_0^{-1}}{\partial \phi_0}$ is simply $\frac{\partial M^{-1}}{\partial \phi}$ evaluated for ϕ_0; similarly, $\frac{\partial N_0^{-1}}{\partial \phi_0}$ is $\frac{\partial N^{-1}}{\partial \phi}$ evaluated for ϕ_0 (use equations (A4)). Equations (A6), along with equations (A2), can now be used to construct equation (4). That is, we now have algebraic formulas for all ten of the partial derivatives needed for equation (4), which was our goal.

Unfortunately, the sailing formulas (A5), and the partial derivatives (A6) derived from them, work properly only for short tracks, or tracks that are nearly east-west. They are not normally adequate for the case we are really interested in: combining observations made over many hours of sailing. For that situation, we need to use more general—and considerably more complicated—sailing formulas. These are given in [12]:

$$\phi = \phi_0 + \frac{M_0}{a(1 - e^2)}\left[\left(1 - \tfrac{3}{4}e^2\right)(\phi' - \phi_0) + \tfrac{3}{8}e^2(\sin 2\phi' - \sin 2\phi_0)\right]$$

$$\lambda = \lambda_0 + \tan C \left(\ln\left[\tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right)\right] + \frac{e}{2}\ln\left[\frac{1 - e\sin\phi}{1 + e\sin\phi}\right] \right. \tag{A7}$$

$$\left. - \ln\left[\tan\left(\frac{\pi}{4} + \frac{\phi_0}{2}\right)\right] - \frac{e}{2}\ln\left[\frac{1 - e\sin\phi_0}{1 + e\sin\phi_0}\right] \right)$$

Note that to use the equation for longitude, λ, we must first evaluate the equation for latitude, ϕ, which, in turn, requires pre-computation of the approximate latitude, ϕ'. Thus, to obtain the required partial derivatives from equations (A7), it is necessary to make liberal use of the chain rule. Since longitude is given as a function of latitude, one of the partial derivatives we will obviously need is

$$\frac{\partial \lambda}{\partial \phi} = \tan C \left[\frac{1}{\cos\phi} - \frac{e^2 \cos\phi}{1 - e^2 \sin^2\phi}\right] \tag{A8}$$

This expression has been considerably simplified from its original form.

The partial derivatives that we need for equation (4) then are:

$$\frac{\partial f}{\partial \phi_0} = \frac{\partial \phi}{\partial \phi_0} = 1 + \frac{1}{a(1 - e^2)}\frac{\partial M_0}{\partial \phi_0}\left[\left(1 - \tfrac{3}{4}e^2\right)(\phi' - \phi_0) + \tfrac{3}{8}e^2(\sin 2\phi' - \sin 2\phi_0)\right]$$

$$+ \frac{M_0}{a(1 - e^2)}\left[\left(1 - \tfrac{3}{4}e^2\right)\left(\frac{\partial \phi'}{\partial \phi_0} - 1\right) + \tfrac{3}{4}e^2\left(\frac{\partial \phi'}{\partial \phi_0}\cos 2\phi' - \cos 2\phi_0\right)\right]$$

$$\frac{\partial f}{\partial \lambda_0} = \frac{\partial \phi}{\partial \lambda_0} = 0$$

$$\frac{\partial f}{\partial C} = \frac{\partial \phi}{\partial C} = \frac{M_0}{a(1 - e^2)}\left[1 - \tfrac{3}{4}e^2 + \tfrac{3}{4}e^2\cos 2\phi'\right]\frac{\partial \phi'}{\partial C}$$

$$\frac{\partial f}{\partial S} = \frac{\partial \phi}{\partial S} = \frac{M_0}{a(1 - e^2)}\left[1 - \tfrac{3}{4}e^2 + \tfrac{3}{4}e^2\cos 2\phi'\right]\frac{\partial \phi'}{\partial S} \tag{A9}$$

$$\frac{\partial g}{\partial \phi_0} = \frac{\partial \lambda}{\partial \phi_0} = -\tan C \left[\frac{1}{\cos\phi_0} - \frac{e^2 \cos\phi_0}{1 - e^2 \sin^2\phi_0}\right] + \frac{\partial \lambda}{\partial \phi}\frac{\partial \phi}{\partial \phi_0}$$

$$\frac{\partial g}{\partial \lambda_0} = \frac{\partial \lambda}{\partial \lambda_0} = 1$$

$$\frac{\partial g}{\partial C} = \frac{\partial \lambda}{\partial C} = \frac{1}{\cos^2 C} \left(\ln \left[\tan \left(\frac{\pi}{4} + \frac{\phi}{2} \right) \right] + \frac{e}{2} \ln \left[\frac{1 - e \sin \phi}{1 + e \sin \phi} \right] \right.$$

$$\left. - \ln \left[\tan \left(\frac{\pi}{4} + \frac{\phi_0}{2} \right) \right] - \frac{e}{2} \ln \left[\frac{1 - e \sin \phi_0}{1 + e \sin \phi_0} \right] \right) + \frac{\partial \lambda}{\partial \phi} \frac{\partial \phi}{\partial C}$$

$$\frac{\partial g}{\partial S} = \frac{\partial \lambda}{\partial S} = \frac{\partial \lambda}{\partial \phi} \frac{\partial \phi}{\partial S}$$

In the first of these expressions, $\frac{\partial M_0}{\partial \phi_0}$ is simply $\frac{\partial M}{\partial \phi}$ evaluated for ϕ_0—see the first of equations (A4). Formulas have already been given for all of the other partial derivatives that are shown unexpanded on the right sides of the expressions above. (This kind of presentation is meant to facilitate the use of these expressions in computer programs. Certain partials and other terms occur repeatedly, and they can be evaluated once, stored, and used wherever needed.) Note that the expression within the first set of brackets in the formula for $\frac{\partial f}{\partial \phi_0}$ is the same as that shown in brackets in the first of equations (A7), and the expression within parentheses in the formula for $\frac{\partial g}{\partial C}$ is the same as that shown in parentheses in the second of equations (A7).

These formulas can be evaluated numerically for each observation, using the best estimates available for the values of all of the quantities involved. Combining these partials with those from equations (A2), we can evaluate the expressions in parentheses in equation (4). That is, we can now construct the conditional equation for each observation, which then enters into the least-squares solution for $\Delta \phi_0, \Delta \lambda_0, \Delta C$, and ΔS.

The $\tan C$ factor in the second of equations (A7) prevents these sailing formulas from being used for east-west courses, or those close to east-west. Equation (A8) shows that the same problem is present in $\frac{\partial \lambda}{\partial \phi}$, which then goes on to serve as a factor in three of the partial derivatives in equations (A9). Therefore, for courses that are within about a half-degree (0.01 radian) of azimuth $90°$ or $270°$, equations (A5) and (A6) should be used. For all other cases, equations (A7), (A8), and (A9) should be used. Note that even in the latter cases, the approximate latitude ϕ' from equation (A5) is needed, as well as the partial derivatives $\frac{\partial \phi'}{\partial \phi_0}$, $\frac{\partial \phi'}{\partial C}$, and $\frac{\partial \phi'}{\partial S}$ from equations (A6).

If all of the angles are expressed in radians, then all of the partial derivatives are dimensionless except those with respect to speed, that is, $\frac{\partial \phi'}{\partial S}$, $\frac{\partial \lambda'}{\partial S}$, $\frac{\partial \phi}{\partial S}$, and $\frac{\partial \lambda}{\partial S}$. These have dimension h/km (that is, inverse speed). That means that the corrections $\Delta \phi_0$, $\Delta \lambda_0$, and ΔC that come out of the least-squares solution will be expressed in radians, and ΔS will be expressed in km/h.

The validity of the formulas for the partials was checked by numerically evaluating them for various points along a large number of rhumb-line tracks and comparing the results at each point with numerical differentiations of the rhumb-line formulas. Despite the crudeness of the numerical differentiation, the two sets of values agreed well; the worse disagreements were in the fifth significant digit. Furthermore, the values of the partials computed from equations (A6) closely matched those from equations (A9) in their region of overlap—that is, for courses with azimuths about a half-degree different from $90°$ or $270°$. There is a certain amount of "overkill" in the above expressions; the region of parameter space within which the problem is linear enough to justify such accuracy is relatively small. We could have approached the entire problem assuming a spherical Earth and obtained much simpler expressions (without terms in e), which would have been quite adequate for celestial navigation as it is now usually practiced. However, with the computing power currently available, there is virtually no penalty in approaching the problem with a better geometric model, and obtaining results that could be used with more accurate automated observing techniques. Furthermore, the increased accuracy does help the problem to converge more rapidly to the correct answer.