New computational approaches to determining the astronomical vessel position based on the Sumner line

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ABSTRACT

In this paper two new approaches are developed to calculate the astronomical vessel position (AVP). Basically, determining the AVP is originated from the spherical equal altitude circles (EACs) concept; therefore, based on the Sumner line's idea, which implies the trial-and-error procedure in assumption, the AVP is determined by using the two proposed approaches. One consists in taking the great circle of spherical geometry to replace the EAC to fix the AVP and the other implements the straight line of the plane geometry to replace the EAC to yield the AVP. To ensure the real AVP, both approaches choose the iteration scheme running in the assumed latitude interval to determine the final AVP. Several benchmark examples are demonstrated to show that the proposed approaches are more accurate and universal as compared with those conventional approaches used in the maritime education or practical operations.

Keywords: celestial navigation; astronomical vessel position; Sumner line; iteration scheme

INTRODUCTION

Daily determination of the astronomical vessel position (AVP) has become a main task for the navigator in every voyage. Although the global positioning system (GPS) has been widely used to determine the vessel position owing to the electronic technology emerging in the modern society, military restriction might arise and lead to its limited commercial application. Therefore, a classical approach to determine the AVP is still necessary because of its reliability or necessity to check results taken from the GPS [3, 4, 25, 28].

To determine the AVP by using the sight reduction method, let's go back to the "initial point" and consider the basic concept of obtaining the AVP, which is originated from the spherical equal altitude circles (EACs). Basically, two celestial bodies produce two EACs whose projections on the chart are called the circles of position (COPs). Some methods developed by using the COP concept to determine the AVP are nearly constructed from the spherical trigonometry. Complex mathematical formulations and tedious calculations arising in these methods make them hard to be used in practical

navigation [1, 2, 5, 7, 10, 14-21, 23, 24, 26, 29-31]. However, when the radius (co-altitude) of the EAC is larger than 3 degrees, which means the curvature of the COP becomes smaller to be neglected, the line of position (LOP) can replace the COP. To date, currently used methods in maritime education or practical operations to determine the AVP included the high-altitude observation (HAO), which is based on the COP concept, and the intercept method (IM), which is based on the LOP concept. The HAO is suitable for the observed altitude greater than 87 degrees and belongs to a kind of graphic method; while the IM is suitable for the observed altitude smaller than 87 degrees and is used to determine the AVP by the graphic method with computations [3-5, 13]. Since the sight reduction methods need to judge the observed altitude for a proper choice, is it possible to develop an approach to determine the AVP without using the graphic method and more accurate and versatile as compared with those methods mentioned above? This is one of our motivations to develop new approaches for yielding the AVP.

The well-known IM or the altitude difference proposed by Marcq de St.-Hilaire (1832-1889) had produced several

types of calculation methods in celestial navigation to obtain the Hilaire line (otherwise called tangent LOP). Then, two tangent LOPs can yield the AVP. Currently used approaches with various short tables or inspection tables for determining the AVP are developed by many researchers [3-6, 11, 13, 20, 21]. However, it should be noted that the original principle of the LOP is the Sumner line (otherwise called the secant LOP), which was rightly called"the commencement of a new era in practical navigation"[3, 4, 22]. Thus, it makes the follow-up approaches be available. The difference between the Sumner line and the Hilaire line consists in their different assumptions. The former is based on two assumed parallels of latitudes to yield two points, which form a Sumner line; while the latter takes an assumed position (AP) to calculate the computed altitude. Then by using the altitude difference between the computed altitude and the observed one, the Hilaire line which moves perpendicularly along the assumed line of azimuth can be obtained. Nevertheless, both schemes of the Sumner LOP and the Hilaire LOP have the same trial-and-error procedure in their assumptions.
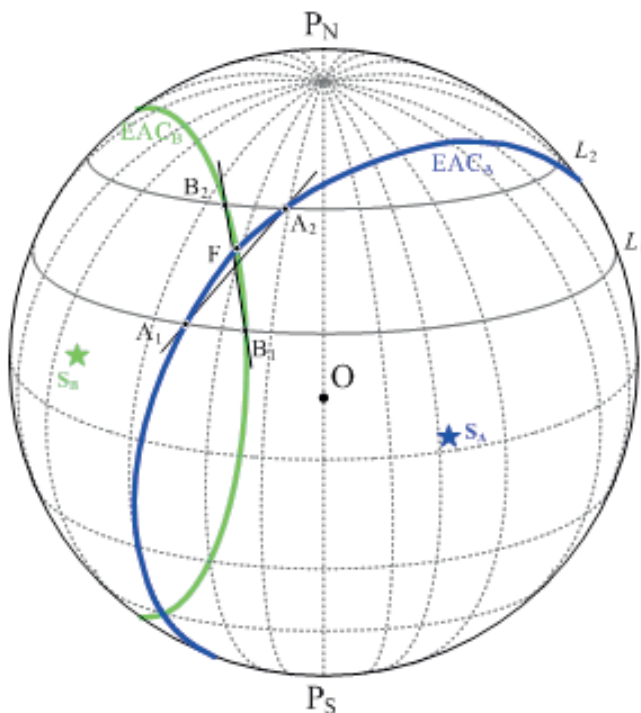

Fig. 1. Illustration of solving the AVP based on the Sumner line's idea

An iteration scheme is then necessary to ensure the real AVP. Since the Sumner line has complicated calculation and limited usage [22], only Valier's method can avoid these shortcomings [31], however its drawbacks still lead to uncertainty of the real AVP.

To overcome drawbacks of the methods developed by using the Sumner line's idea, the iteration scheme in conjunction with the limit concept is assumed to develop our approaches for accuracy and versatility. As known, determining the AVP comes from the EACs concept. In this regard, based on the Sumner line's idea, the AVP can be obtained either by taking

the great circle of spherical geometry to replace the EAC or by adopting the straight line of the plane geometry to replace the EAC to determine the AVP. The two our approaches are thus developed to yield the real AVP with avoiding application of graphic construction and tedious calculation. In addition, the iteration scheme and the limit concept make the two approaches simpler and more easily understandable. The principles used in the proposed approaches are illustrated in the following sections.

Section 2 of this paper presents mathematical background and related formulae of the proposed approaches. Calculation procedures of the two approaches are illustrated in Section 3. Section 4 demonstrates validation examples. Finally, concrete conclusions are presented in Section 5.
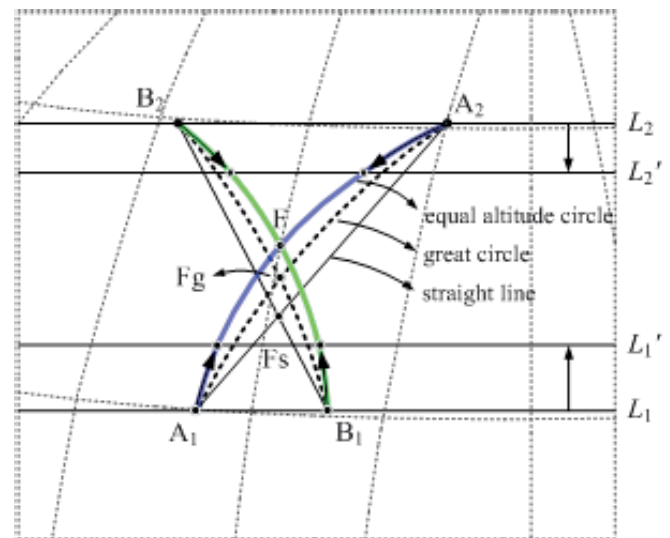

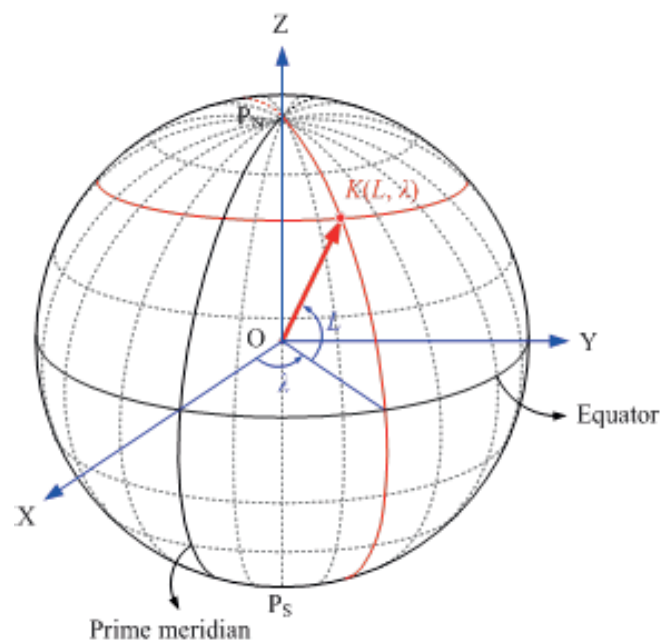Fig. 2. Illustration of iteration scheme

Fig. 3. Transformation of the spherical coordinates system into the Cartesian coordinates system

THEORETICAL BACKGROUND

As shown in Fig. 1, the idea of the Sumner line consists in selecting two parallels of latitudes around the dead reckoning (DR) to have two intersections with the EAC of celestial body. Captain Thomas Hubbard Sumner (1807-1876) implemented the line (called LOP) formed by the two points and projected it on the Mercator chart. Since the mathematical procedure constructed by Captain Sumner was rather tedious and cumbersome to use, Valier's method was proposed to project the LOP on the Cartesian coordinates system for simplifying the determination of the AVP [22, 31]. However, this method is too rough to obtain the AVP accurately. Because the Sumner line has a trial-and-error characteristic, the iteration scheme is used to resolve the shortcoming of this characteristic. Since the limit concept is available, two points on the EAC can be replaced either by a great circle of spherical geometry or a straight line of the plane geometry as shown in Fig. 2. It should be noticed that the four intersections are moving along the EACs. Derivations of the formulae used in the spherical and plane geometry are illustrated below. All symbols used in this paper are listed in Nomenclature.

Derivation of the formulae used in the spherical geometry

We first treat the Earth as a unitary sphere. From the viewpoint of the navigator, the Earth coordinate system can replace the spherical coordinates system. In this regard, the vector for any point K on the Earth's surface shown in Fig. 3 can be represented with the latitude „L" and longitude „λ" in a Cartesian coordinates system as follows :

$$\vec{K}=[\cos L \cos\lambda,\ \cos L \sin\lambda,\ \sin L],\ L=\left[-\frac{\pi}{2}, \frac{\pi}{2}\right],\ \lambda=[0, 2\pi) \quad (1)$$

imilarly, we also treat the celestial sphere as a unitary sphere. An astronomical triangle is constructed by the Earth, celestial equator and celestial horizon systems of coordinates. To increase the solving efficiency and simplify the used formulae, the fixed coordinate system and relative celestial meridian concept in conjunction with vector algebra are applied to derive the related formulae. As shown in Fig. 4, every point of the spherical EAC, the elevated pole (P) and the celestial body ("S") form an astronomical triangle $\hat{\Delta}FPS"$. Position vectors and their dot product are illustrated in the following.

$$\vec{F}=[\ \cos L_F,\ 0,\ \sin L_F\ ] \quad (2a)$$

$$\vec{S}=[\ \cos d \cos t,\ \cos d \sin t,\ \sin d\ ] \quad (2b)$$

$$\vec{F} \bullet \vec{S} = \cos(zd) = \sin H = \\ = \cos L_F \cos d \cos t + \sin L_F \sin d \quad (2c)$$

Equation (2c) is the well-known side cosine law in spherical trigonometry. Once the declination (d) and the observed altitude (H) of the celestial body as well as the latitude (L_F) of the AVP are given, the longitude ($λ_F$) of the AVP on the spherical EAC can be easily yielded.
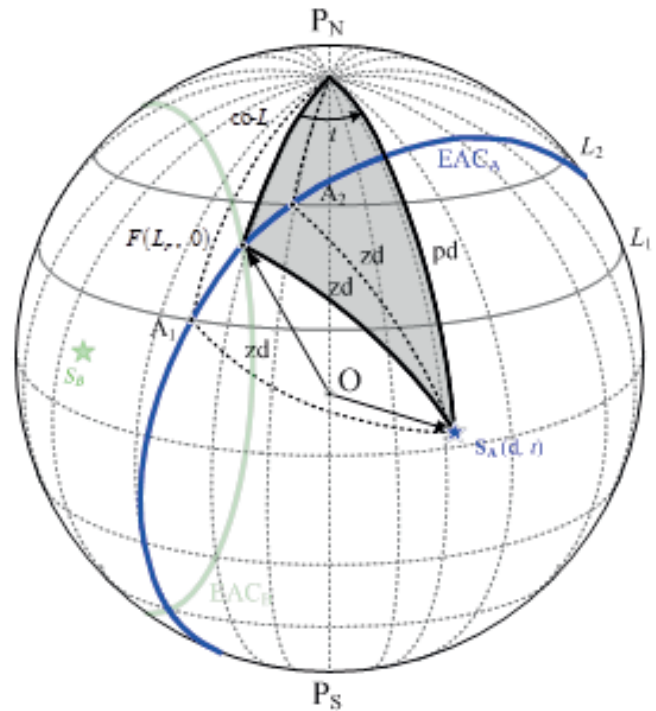

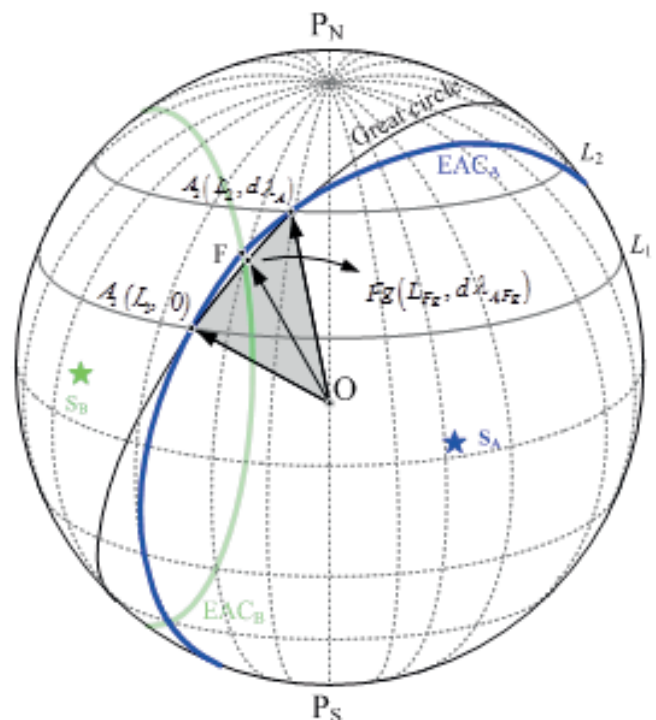Fig. 4. An astronomical triangle in celestial navigation

Fig. 5. Three vectors of the great circle equation in spherical geometry

$$\cos t = \frac{\sin H - \sin L_F \sin d}{\cos L_F \cos d} \quad (3a)$$

$$\lambda_F = G \mp t \ (W/E) \quad (3b)$$

Because the real AVP is unknown, the two parallels, L_1 and L_2, are assumed to have two intersections with the EAC of celestial body A, that is, A_1 and A_2, as shown in Fig. 4.

The related formulae are illustrated as follows.

$$L_{1,2} = L_{DR} \mp 10' \qquad (4a)$$

$$\cos(t_{1,2}) = \frac{\sin H_A - \sin(L_{1,2}) \sin d_A}{\cos(L_{1,2}) \cos d_A} \qquad (4b)$$

$$\lambda_{A_1, A_2} = G_A \mp (t_{1,2}) \qquad (4c)$$

Similarly, the two parallels, L_1 and L_2, are assumed to have two intersections with the EAC of celestial body B, that is B_1 and B_2, the related formulae are shown in the following.

$$L_{1,2} = L_{DR} \mp 10' \qquad (5a)$$

$$\cos(t_{1,2}) = \frac{\sin H_B - \sin(L_{1,2}) \sin d_B}{\cos(L_{1,2}) \cos d_B} \qquad (5b)$$

$$\lambda_{B_1, B_2} = G_B \mp (t_{1,2}) \qquad (5c)$$

It should be noticed that $\mp 10'$ term of the equations (4a) and (5a) is suggested by [12]. As shown in Fig. 5, three points, AVP, A_1 and A_2, are located in the small circle, that is, EAC of the celestial body A. Since the small circle equation varies with the different observed altitude, one takes the great circle to replace the small circle for simplifying the calculation. Then, A_1, A_2 and F_g can form a great circle taken from the centre of the Earth. Because the three vectors are coplanar, the scalar triple product is equal to zero [27]. Their vector formulation in the Cartesian coordinates system can be expressed as follows:

$$(\overrightarrow{A_1} \times \overrightarrow{A_2}) \bullet \overrightarrow{Fg} = 0 \qquad (6)$$

Components of the parameter vector $A_1 \times A_2$ for the celestial body A can be expressed as follows:

$$a_1 = -\sin L_1 \cos L_2 \sin d\lambda_A \qquad (7a)$$

$$a_2 = \sin L_1 \cos L_2 \cos d\lambda_A - \cos L_1 \sin L_2 \qquad (7b)$$

$$a_3 = \cos L_1 \cos L_2 \sin d\lambda_A \qquad (7c)$$

and the great circle equation (GCE) [5, 8, 9] :

$$a_1 \cos L_{Fg} \cos d\lambda_{A_1Fg} + a_2 \cos L_{Fg} \sin d\lambda_{A_1Fg} + \\ + a_3 \sin L_{Fg} = 0 \qquad (8)$$

Once the longitude increment ($d\lambda_{A_1Fg}$) is given, the latitude (L_{Fg}) can be easily obtained. Hence, rearranging the equation (8), one can yield:

$$\tan L_{Fg} = \frac{a_1 \cos d\lambda_{A_1Fg} + a_2 \sin d\lambda_{A_1Fg}}{-a_3} \qquad (9)$$

Similarly, components of the parameter vector for the celestial body B and the GCE are illustrated, respectively, as follows.

$$b_1 = -\sin L_1 \cos L_2 \sin d\lambda_B \qquad (10a)$$

$$b_2 = \sin L_1 \cos L_2 \cos d\lambda_B - \cos L_1 \sin L_2 \qquad (10a)$$

$$b_3 = \cos L_1 \cos L_2 \sin d\lambda_B \qquad (10a)$$

and

$$\tan L_{Fg} = \frac{b_1 \cos d\lambda_{B_1Fg} + b_2 \sin d\lambda_{B_1Fg}}{-b_3} \qquad (11)$$

Derivation of the formulae used in the plane geometry

When the Sumner LOP is projected on the Mercator chart, the calculation is too complicated to use by the navigator. Due to the conformal transformation existing in the rectangular coordinate system and the Mercator chart, one can replace the Mercator chart by the rectangular coordinate system for simplicity. Therefore, position variables of the Earth coordinate system, latitude and longitude, can be put in the Cartesian coordinates system for calculation, that is, the longitude and latitude can replace the values of the x-axis and y-axis.

• Geometrical analysis method (Valier's method)

It is easy to prove that both the latitude increment and longitude increment can be derived by using the property of similar triangles and proportion by addition. Let's first consider the celestial body A as a viewpoint to derive the related formulae as follows.

1. Latitude increment (dL_{Fs} or $\overline{P_1Fs}$) as shown in Fig. 6
Since $\Delta A_1B_1Fs \sim \Delta A_2B_2Fs$, their corresponding altitudes have the same ratio as a pair of corresponding sides, that is:

$$\overline{P_1Fs} : \overline{P_2Fs} = \overline{A_1B_1} : \overline{A_2B_2} \qquad (12)$$

By considering the property of proportion by addition, one achieves:

$$\overline{P_1Fs} : \overline{P_1P_2} = \overline{A_1B_1} : (\overline{A_1B_1} + \overline{A_2B_2}) \qquad (13)$$

Rearranging the equation (13) yields

$$\overline{P_1Fs} = \left(\frac{\overline{A_1B_1}}{\overline{A_1B_1} + \overline{A_2B_2}}\right) \overline{P_1P_2} \qquad (14)$$

Longitude increment ($d\lambda_{A_1Fs}$ or $\overline{HFs}$) as shown in Fig. 6. Since $\Delta A1HFs \sim \Delta A1QA2$, their corresponding altitudes have the same ratio as a pair of corresponding sides, that is:

$$\overline{HFs} : \overline{QA_2} = \overline{A_1H} : \overline{A_1Q} \qquad (15)$$

By considering the property of proportion by addition and equation (13), one obtains:

$$\overline{\text{HFs}} : \overline{\text{QA}_2} = \overline{\text{P}_1\text{Fs}} : \overline{\text{P}_1\text{P}_2} = \overline{\text{A}_1\text{B}_1} : (\overline{\text{A}_1\text{B}_1} + \overline{\text{A}_2\text{B}_2}) \qquad (16)$$

Rearranging the equation (16) yields:

$$\overline{\text{HFs}} = \left(\frac{\overline{\text{A}_1\text{B}_1}}{\overline{\text{A}_1\text{B}_1} + \overline{\text{A}_2\text{B}_2}} \right) \overline{\text{QA}_2} \qquad (17)$$

By observing the equations (14) and (17) as well as replacing the rectangular coordinate system with the Earth coordinate system, one can find the following equation:

$$r = \frac{\overline{\text{A}_1\text{B}_1}}{\overline{\text{A}_1\text{B}_1} + \overline{\text{A}_2\text{B}_2}} = \frac{\left| \lambda_{A_1} - \lambda_{B_1} \right|}{\left| \lambda_{A_1} - \lambda_{B_1} \right| + \left| \lambda_{A_2} - \lambda_{B_2} \right|} \qquad (18)$$

in which r is the positive increment ratio. Then, the equations (14) and (17) can be rewritten respectively as follows:

$$dL_{Fs} = r\, dL \qquad (19a)$$

$$d\lambda_{A_1Fs} = r\, d\lambda_A \qquad (19b)$$

Rearranging the equations (19a) and (19b) yields the intersection of two Sumner LOPs as follows:

$$L_{Fs} = L_1 + r\, dL \qquad (20a)$$

$$\lambda_{Fs} = \lambda_{A_1} + r\, d\lambda_A \qquad (20b)$$

Tab. 1. *Needed information for solving the AVP in example 1*

Celestial body	Kochab	Spica
ZT, DR	20-11-26, (39°00.0′ N, 157°10.0′ W)	
Observed ZT	20-07-43	20-11-26
H	47°13.6′	32°28.7′
G	463°43.0′	486°05.7′
d	74°10.6′ N	11°08.4′ S

Source: [4] *American Practical Navigator*, pp. 301-303 (2002)
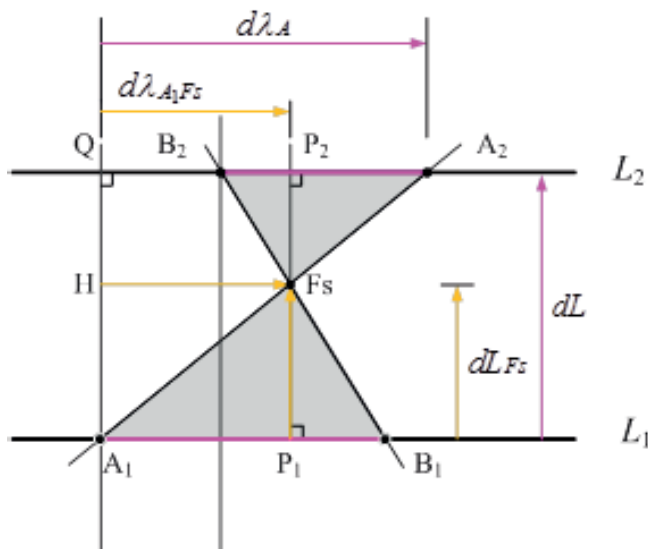

Fig. 6. Similar triangles in plane geometry

Similarly, by considering the celestial body B as a viewpoint, the related formulae are yielded as follows :

$$dL_{Fs} = r\, dL \qquad (21a)$$

$$d\lambda_{B_1Fs} = r\, d\lambda_B \qquad (21b)$$

Rearranging the equations (21a) and (21b) yields the intersection of two Sumner LOPs as follows:

$$L_{Fs} = L_1 + r\, dL \qquad (22a)$$

$$\lambda_{Fs} = \lambda_{B_1} + r\, d\lambda_B \qquad (22b)$$

- Algebraic equations method (simultaneous linear equations)

Based on the algebra, the linear equations for Sumner LOPs of the celestial bodies A and B can be formulated as follows:

$$(d\lambda_A)L_{Fs} - (dL)\lambda_{Fs} = d\lambda_A\, L_1 - dL\, \lambda_{A_1} \qquad (23a)$$

and

$$(d\lambda_B)L_{Fs} - (dL)\lambda_{Fs} = d\lambda_B\, L_1 - dL\, \lambda_{B_1} \qquad (23b)$$

By using the Cramer's rule the solutions of the above equations are :

$$L_{Fs} = L_1 + r\, dL \qquad (24a)$$

and

$$\lambda_{Fs} = \lambda_{A_1} + r\, d\lambda_A = \lambda_{B_1} + r\, d\lambda_B \qquad (24b)$$

In fact, the equation (24b) is a combination of the equations (20b) and (22b). It is found that the AVP result obtained from geometrical analysis method is the same as that from algebraic equations method.

It should be noticed that the longitude increments for the celestial body A ($d\lambda_{A1Fg}$) and body B ($d\lambda_{B1Fg}$) of the great circle are hard to be obtained; therefore, the longitude increments of the great circle needs to be replaced by those of the straight line, that is, $d\lambda_{A1Fs}$ and $d\lambda_{B1Fs}$. Consequently, the equations (9) and (11) can be rewritten as follows:

$$\tan L_{Fg} = \frac{a_1 \cos d\lambda_{A_1Fs} + a_2 \sin d\lambda_{A_1Fs}}{-a_3}$$

$$\tan L_{Fg} = \frac{b_1 \cos d\lambda_{B_1Fs} + b_2 \sin d\lambda_{B_1Fs}}{-b_3}$$

CONSTRUCTING THE SOLVING PROCEDURES

In this section, determining the AVP by taking the great circle of spherical geometry to replace the EAC is called the Sumner-GC approach, while using the straight line of the plane geometry to replace the EAC is called the Sumner-SL approach.

Sumner-GC approach

Step 1. The four intersections can be obtained by using equations (4a), (4b) and (4c) for A_1 and A_2, while using the equations (5a), (5b) and (5c) for B_1 and B_2.

Step 2. The longitude of the possible AVP from the celestial body A is determined by using the equations (18) and (20b) or that from the celestial body B is yielded by using the equations (18) and (22b).

Step 3. The latitude of the possible AVP from the celestial body A is determined by using the equations (19b) and (25) or that from the celestial body B can be yielded by using the equations (21b) and (26).

Step 4. The iteration scheme is introduced to reach the real AVP if the AVPs calculated from the celestial bodies A and B are different and the real AVP is not ensured, then the iterating step 1 to step 3 is necessary.

Sumner-SL approach

Step 1. The four intersections can be obtained by using the equations (4a), (4b) and (4c) for A_1 and A_2, while by using the equations (5a), (5b) and (5c) for B_1 and B_2.

Step 2. The latitude of the possible AVP is determined by using the equations (18) and (24a).

Step 3. The longitudes of the possible AVP from the celestial body A or celestial body B are determined by using the equations (18) and (24b), respectively.

Step 4. The real AVP is not ensured until the iterating step 1 to step 3 is performed.

VALIDATION EXAMPLES AND DISCUSSIONS

Validation examples

Example 1. The 2011 DR position of a vessel is" L 39°00.0' N, λ 157°10.0' W". At 20-11-26, the star Spica is observed with a sextant. At 20-07-43, shortly before the above observation, another star, the Kochab is spotted. The navigator records the needed information and further reduces it from the nautical almanac for sight reduction as shown in Tab. 1 (Bowditch, 2002. pp.301-303).

Tab. 2. Detailed solving procedures used in the Sumner-GC approach for example 1

Celestial body	Eq.	Kochab (S_A)	Spica(S_B)	Eq.
L_1	(4a)	38°50.0' N		(5a)
L_2	(4a)	39°10.0' N		(5a)
t_1	(4b)	52°00.4' E	30°33.6' E	(5b)
t_2	(4b)	53°16.8' E	29°58.0' E	(5b)
$A_1; B_1$	(4c)	(38°50.0' N, 155°43.4' W)	(38°50.0' N, 156°39.3' W)	(5c)
$A_2; B_2$	(4c)	(39°10.0' N, 156°59.8' W)	(39°10.0' N, 156°03.7' W)	(5c)
r	(18)	0.4993	0.4993	(18)
$d\lambda_{A_1Fs}; d\lambda_{B_1Fs}$	(19b)	0°38.2' W	0°17.8' E	(21b)
F(0)	(20b) (25)	(39°00.1' N, 156°21.5' W)	(39°00.0' N, 156°21.5' W)	(22b) (26)
F(1)	(20b) (25)	(39°00.0' N, 156°21.7' W)	(39°00.0' N, 156°21.7' W)	(22b) (26)
F(2)	(20b) (25)	(39°00.0' N, 156°21.7' W)	(39°00.0' N, 156°21.7' W)	(22b) (26)
Fix[*]		(39°00.0' N, 156°21.7' W)		

[*]: The true AVP has been validated in [7]

Tab. 3. Detailed solving procedures used in the Sumner-SL approach for example 1

Celestial body	Eq.	Kochab (S_A)	Spica(S_B)	Eq.
L_1	(4a)	38°50.0' N		(5a)
L_2	(4a)	39°10.0' N		(5a)
t_1	(4b)	52°00.4' E	30°33.6' E	(5b)
t_2	(4b)	53°16.8' E	29°58.0' E	(5b)
$A_1; B_1$	(4c)	(38°50.0' N, 155°43.4' W)	(38°50.0' N, 156°39.3' W)	(5c)
$A_2; B_2$	(4c)	(39°10.0' N, 156°59.8' W)	(39°10.0' N, 156°03.7' W)	(5c)
r	(18)	0.4993	0.4993	(18)
F(0)	(24a) (24b)	(39°00.0' N, 156°21.5' W)	(39°00.0' N, 156°21.5' W)	(24a) (24b)
F(1)	(24a) (24b)	(39°00.0' N, 156°21.7' W)	(39°00.0' N, 156°21.7' W)	(24a) (24b)
F(2)	(24a) (24b)	(39°00.0' N, 156°21.7' W)	(39°00.0' N, 156°21.7' W)	(24a) (24b)
Fix[*]		(39°00.0' N, 156°21.7' W)		

[*]: The true AVP has been validated in [7]

Available methods: The AVP can be determined by the proposed Sumner-GC approach and Sumner-SL approach.

Solution:
1. By using the Sumner-GC approach with the iteration scheme, the AVP, L 39°00.0' N, λ 156°21.7' W, can be determined without plotting. Results and detailed solving procedures are listed in Tab. 2.
2. By using the Sumner-SL approach with the iteration scheme, the AVP, L 39°00.0' N, λ 156°21.7' W, can be determined without plotting. Results and detailed solving procedures are listed in Tab. 3.

Remark: The true AVP is pointed at L 39°00.0 N, λ 156°21.7' W and has been validated in [7]. In general, the distortion of the EAC at low observed altitude will be larger than that at high observed altitude; however solutions of our two approaches with the iteration scheme for this example are both the same as the true AVP. It means that the proposed approaches are accurate and verified through this example.

Example 2. The 1224 DR position of a vessel is L 20°17.4' N, λ 50°07.4' W. The ship in on course 127° at speed of 18

knots. The navigator observes the lower limb of the Sun twice. The first observation is made at 12-15-15. The second observation is made at 12-24-13. The navigator records the needed information and further reduces it from the nautical almanac for sight reduction as shown in Tab. 4 [3].

Available methods: The AVP can be determined by the proposed Sumner-GC approach (approach 1) and Sumner-SL approach (approach 2), further plotting the graphic AVP on the universe plotting sheet (UPS) to illustrate iteration scheme.

Tab. 4. Needed information for solving the AVP in example 2

Celestial body	Sun	Sun
ZT, DR	1224, (20°17.4′ N, 50°07.4′ W)	
Observed ZT	12-15-15	12-24-13
H	88°09.2′	87°42.8′
G	49°25.6′	51°40.1′
d	21°53.1′ N	21°53.1′ N

Solution:
1. By using the Sumner-GC approach with the iteration scheme, the AVP, L 20°08.0' N, λ 50°05.7' W, can be determined without plotting. Results and detailed solving procedures are listed in Tab. 5 and the graphic AVP on the UPS is shown in Fig. 7.
2. By using the Sumner-SL approach with the iteration scheme the AVP, L 20°08.0' N, λ 50°05.7' W, can be determined without plotting. Results and detailed solving procedures are listed in Tab. 6 and the graphic AVP on the UPS is shown in Fig. 7.

Tab. 5. Detailed solving procedures used in the Sumner-GC approach for example 2

Celestial body	Eq.	Sun (S_A)	Sun (S_B)	Eq.
L_1	(4a)	20°07.4′ N		(5a)
L_2	(4a)	20°27.4′ N		(5a)
t_1	(4b)	0°40.6′ E	1°33.7′ W	(5b)
t_2	(4b)	1°17.4′ E	1°54.9′ W	(5b)
$A_1; B_1$	(4c)	(20°07.4′ N, 50°04.0′ W)	(20°07.4′ N, 50°06.4′ W)	(5c)
$A_2; B_2$	(4c)	(20°27.4′ N, 50°40.8′ W)	(20°27.4′ N, 49°45.2′ W)	(5c)
r	(18)	0.0406	0.0406	(18)
$d\lambda_{A_1Fs}; d\lambda_{B_1Fs}$	(19b)	0°01.5′ W	0°00.9′ E	(21b)
F(0)	(20b) (25)	(20°08.2′ N, 50°05.5′ W)	(20°08.2′ N, 50°05.5′ W)	(22b) (26)
F(4)	(20b) (25)	(20°08.0′ N, 50°05.7′ W)	(20°08.0′ N, 50°05.7′ W)	(22b) (26)
Fix		(20°08.0′ N, 50°05.7′ W)		

Tab. 6. Detailed solving procedures used in the Sumner-SL approach for example 2

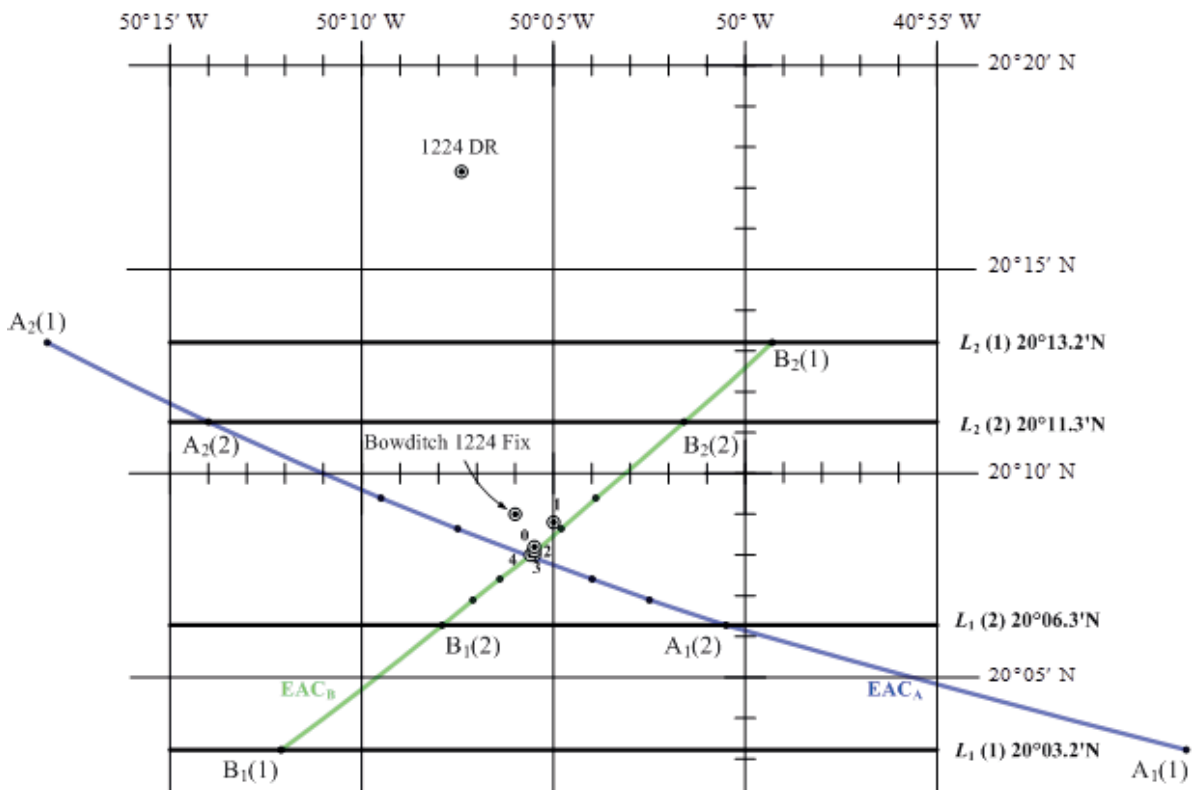| Celestial body	Eq.	Sun (S_A)	Sun (S_B)	Eq.
L_1	(4a)	20°07.4′ N		(5a)
L_2	(4a)	20°27.4′ N		(5a)
t_1	(4b)	0°40.6′ E	1°33.7′ W	(5b)
t_2	(4b)	1°17.4′ E	1°54.9′ W	(5b)
$A_1; B_1$	(4c)	(20°07.4′ N, 50°04.0′ W)	(20°07.4′ N, 50°06.4′ W)	(5c)
$A_2; B_2$	(4c)	(20°27.4′ N, 50°40.8′ W)	(20°27.4′ N, 49°45.2′ W)	(5c)
r	(18)	0.0406	0.0406	(18)
F(0)	(24a) (24b)	(20°08.2′ N, 50°05.5′ W)	(20°08.2′ N, 50°05.5′ W)	(24a) (24b)
F(4)	(24a) (24b)	(20°08.0′ N, 50°05.7′ W)	(20°08.0′ N, 50°05.7′ W)	(24a) (24b)
Fix		(20°08.0′ N, 50°05.7′ W)		

Fig. 7. Illustrations of the AVP with iteration scheme on the UPS in example 2

Tab. 7. Iteration results by using the approach 1 and approach 2 for example 2

Iteration	dL	r*	Approach 1	r	Approach2
F(0)	20′	0.0406	20°08.2′ N 50°05.5′ W	0.0406	20°08.2′ N 50°05.5′ W
F(1)	10′	0.5549	20°08.8′ N 50°05.0′ W	0.5553	20°08.8′ N 50°05.0′ W
F(2)	5′	0.3694	20°08.1′ N 50°05.5′ W	0.3721	20°08.1′ N 50°05.5′ W
F(3)	2.5′	0.4527	20°08.0′ N 50°05.6′ W	0.4535	20°08.0′ N 50°05.6′ W
F(4)	1.25′	0.4724	20°08.0′N 50°05.7′W	0.4729	20°08.0′N 50°05.7′W

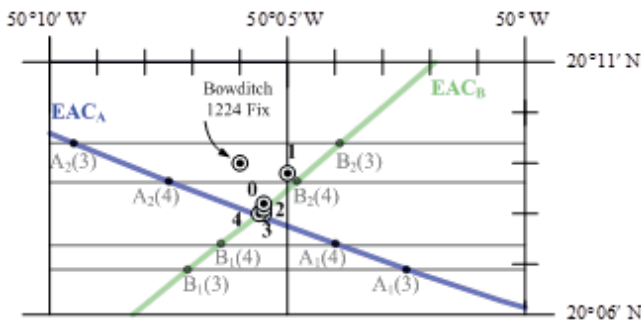*: The positive increment ratio is obtained from the celestial body A

Fig. 8. Central part of Fig. 7 in larger scale

Remark: The solved AVP in [3] is L 20°09.0' N, λ 50°06.0' W. This solution should be inaccurate because the EAC projection on Mercator chart will be distorted. Since example 1 has validated our two approaches and both of approaches 1 and 2 yield the same solution for the AVP in this example, the true AVP can be ensured. As shown in Tab. 7 and Fig. 8, the iteration numbers of the two approaches are the same. It means that the great circle and the straight line are very close to each other in the small area of spherical surface. Besides, it is found that when the positive increment ratio (r) approaches to 0.5, the result of the iteration scheme will reach true AVP. In summary, both approaches are more versatile and accurate than those conventional methods used in the maritime education or practical operation.

Although the proposed approaches are originated from the Sumner line's idea, which had been formulated a long time ago, yet the iteration scheme can be adopted in the solving procedures as the computation technology develops nowadays. After conducting this research work, we state that originality of the past valuable concept might find its useful applications, depending on how we make use of presently developed technology.

CONCLUSIONS

In this paper, based on the Sumner line's idea, two approaches have been developed to determine the AVP. By way of assumed latitude interval, intersections of two straight lines and two great circles formulate the schemes of the two approaches, respectively. Due to the trial-and-error characteristic of the Sumner line's idea, the iteration scheme and the limit concept are implemented to reach the real AVP. The calculation procedures of the two approaches are also adjusted to practical usage. Several validation examples have verified the proposed approaches successfully. It is shown that the new approaches can determine the real AVP more versatile and accurate than those obtained from conventional methods, the HAO and the IM, used in the maritime education or practical operation.

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