Astron. Astrophys. 202, 309–315 (1988)

ASTRONOMY AND ASTROPHYSICS

Planetary theories in rectangular and spherical variables. VSOP 87 solutions

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Received November 20, 1987; accepted March 23, 1988

Summary. Up to this time, the VSOP (Variations Séculaires des Orbites Planétaires) analytical solutions of the motion of the planets were only represented in elliptic variables, but the cartesian or spherical variables are much more convenient in many problems: determination of the planetary perturbations of the Moon, analytical expressions for the computation of the apparent places, analytical expressions of nutation, of the difference TDB–TDT.

From an analytical solution of the motion expressed with elliptic elements, we hence build different representations. The solutions are expressed with rectangular variables X, Y, Z or with spherical variables, longitude, latitude and radius vector. The different reference frames used are the dynamical ecliptic and equinox J2000.0, the ecliptic and equinox of date. The origin is the Sun or the barycenter of the solar system.

With these constructions we give the algorithms which allow to go from the elliptic variables to the rectangular variables and from the rectangular variables to the spherical variables in the case of analytical solutions of the time. We also give the precession matrix as a function of the time to be able to hand over coordinates from the reference frame J2000.0 to the reference frame of date. This matrix can be used on a time span of several thousands of years before and after J2000.0

The different versions are available on magnetic tape. They are accompanied with Fortran programs substituting time in the series and computing the derivatives with respect to time.

Key words: celestial mechanics – planetary theory

1. Introduction

Up to now the analytical solutions VSOP (Variations Séculaires des Orbites Planétaires) had only been represented in elliptic variables. It happens that, for some problems, these variables are less convenient than the cartesian or spherical coordinates. It is so for example in the analytical construction of the planetary perturbations of the Moon (Chapront-Touzé and Chapront, 1983), in the determination of analytical expressions for the calculation of apparent places (Soma et al., 1988), (Ron and Vondrak, 1986). It is also true for the construction of analytical expressions of nutation, of the discrepancy between the barycentric dynamical time (TDB) and the terrestrial dynamical time (TDT). We hence constructed, with all the precision of the basic solution, various forms of an analytical solution of the planetary motion coming from the VSOP 82 solution (Bretagnon, 1982). Let us remind that the VSOP 82 solution is a theory of the motion of the planets from Mercury to Neptune the integration constants of which have been determined by fitting to the DE200 numerical integration of the J.P.L. (Standish, 1982). VSOP 82 contains the whole of the perturbations up to the third order of the masses for all the planets. For the outer planets, Jupiter, Saturn, Uranus and Neptune, the solution is completed up to the sixth order of the masses by an iterative method.

We have built a VSOP 87 solution in elliptic elements close to VSOP 82. From VSOP 87, we have made up five representations (VSOP 87A–B–C–D–E). These various versions are different from one to another in the type of coordinates (rectangular or spherical), the epoch and reference frame used (ecliptic and equinox J2000.0 or of date).

In the same way as for VSOP 82, the VSOP 87 solution refers to inertial and dynamical ecliptic and equinox J2000.0. This reference frame is linked to that of the numerical integration DE200 by the relations:

$$\gamma^{(R)}_{200}\gamma^{(I)}_D = -0.0930'' \text{ reckoned in the equator}$$

$$\varepsilon^{(I)}_D = 23°26'21.409\,1'' \quad (1)$$

where $\gamma^{(I)}_D$, $\varepsilon^{(I)}_D$ represent the dynamical equinox and the dynamical obliquity when meaning inertial.

Standish (1981) defines the rotating dynamical equinox $\gamma^{(R)}_D$ and the obliquity $\varepsilon^{(R)}_D$ by:

$$\gamma^{(R)}_D\gamma^{(I)}_D = -0.093\,66''$$

$$\varepsilon^{(R)}_D - \varepsilon^{(I)}_D = +0.003\,34''$$

hence:

$$\gamma^{(R)}_{200}\,\gamma^{(R)}_D = +0.000\,66''$$

The rotating dynamical equinox of the solutions VSOP is thus very close to the equinox of DE200.

Let us recall that the determination of the equinox $\gamma^{(R)}$ of DE118 and hence of DE200 is according to Standish (1982b) remarkably closely with that of Fricke used for the construction of the FK5.

The bodies which motion has been represented can be different according to the solutions. We give in Table 1 the list of these bodies for the VSOP 87 solution and the deriving solutions as well as the precision for each body. The Earth-Moon barycenter has been called EMB.

Table 1. VSOP 87 solutions. Precision over the time-span 1900–2100

		Bodies								
Solution	Variables	1	2	3	4	5	6	7	8	9
VSOP 87	a, λ, k, h, q, p	Me	V	EMB	M	J	S	U	N	
VSOP 87A	X, Y, Z	Me	V	E	M	J	S	U	N	EMB
VSOP 87B	L, B, r	Me	V	E	M	J	S	U	N	
VSOP 87C	X, Y, Z	Me	V	E	M	J	S	U	N	
VSOP 87D	L, B, r	Me	V	E	M	J	S	U	N	
VSOP 87E	X, Y, Z	Me	V	E	M	J	S	U	N	Sun
Precision		0.001″	0.006″	0.005″	0.023″	0.020″	0.100″	0.016″	0.030″	

In this paper, after having mentioned our notations and given some characteristics common to our solutions, we will describe the VSOP 87 solution and its different versions as well as the way through which they have been built.

2. Notations. Characteristics common to the different versions

We use the elliptic variables a, λ, k, h, q, p with:

$$k = e\cos\varpi; \quad q = \sin\frac{i}{2}\cos\Omega$$

$$h = e\sin\varpi; \quad p = \sin\frac{i}{2}\sin\Omega.$$

where a is the semi-major axis, λ the mean longitude, e the eccentricity of the orbit, ϖ the longitude of the perihelion, i the inclination, Ω the longitude of the node.

We call:

X, Y, Z the rectangular coordinates,

L, B, r the spherical coordinates.

In our solutions the coordinates are measured in au for what regards the lengths (X, Y, Z, a, r) and in radians for the other quantities. These coordinates are explicit functions of time and are under the form of periodic series and Poisson series. Each term is given under two forms:

$$T^{\alpha}(S\sin\varphi + K\cos\varphi) = T^{\alpha}A\cos(B + CT) \qquad (2)$$

In the expressions (2):

—the time T is reckoned in thousands of Julian years from J2000.0

$$T = (\text{Julian date } -2\,451\,545)/365\,250$$

—the power α of the time T is an integer in-between 0 and 5.

—the argument φ is defined by:

$$\varphi = \sum_{i=1}^{12} a_i\lambda_i$$

where the a_i are integers.

The quantities λ_i, for $i = 1$ to 8, represent the mean longitudes of the eight planets. For $i = 9$, 10, 11, and λ_i are Delaunay arguments of the Moon D, F, L, respectively. Finally $\lambda_{12} = \zeta$ represents the mean longitude of the Moon given with respect to the equinox of date.

Each λ_i is under the form:

$$\lambda_i = \lambda_i^o + N_i T$$

The λ_i^o and N_i are given in Table 2.

The two forms of formula (2) are linked by the following relations:

$$A = \sqrt{S^2 + K^2}; \quad B = \sum_{i=1}^{12} a_i\lambda_i^o + \beta; \quad C = \sum_{i=1}^{12} a_i N_i.$$

where β is given by:

$$S = -A\sin\beta; \quad K = A\cos\beta.$$

The series are organized according to $|S| + |K|$ decreasing which allows to truncate them depending on the precision wanted.

Let us note that since the coordinates are explicit functions of the time, it is easy to get the derivatives with respect to time, and thus the velocities.

$$\frac{d}{dt}[T^{\alpha}(S\sin\varphi + K\cos\varphi)] = \alpha T^{\alpha-1}(S\sin\varphi + K\cos\varphi) + T^{\alpha}(-K\sin\varphi + S\cos\varphi) \times \sum_{i=1}^{12} a_i N_i = \alpha T^{\alpha-1}A\cos(B + CT) - T^{\alpha}AC\sin(B + CT) \qquad (3)$$

Table 2. Phases λ_i^0 (in radians) and frequencies N_i (in radians per thousand Julian years) of the 12 components of the arguments

i	λ_i^0	N_i
1	4.402 608 842 40	26 087.903 141 574 2
2	3.176 146 696 89	10 213.285 546 211 0
3	1.753 470 459 53	6 283.075 849 991 4
4	6.203 476 112 91	3 340.612 426 699 8
5	0.599 546 497 39	529.690 965 094 6
6	0.874 016 756 50	213.299 095 438 0
7	5.481 293 871 59	74.781 598 567 3
8	5.311 886 286 76	38.133 035 637 8
9	5.198 466 741 03	777 13.771 468 120 5
10	1.627 905 233 37	84 334.661 581 308 3
11	2.355 555 898 27	83 286.914 269 553 6
12	3.810 344 546 97	83 997.091 135 595 4

3. VSOP 87 solution

This solution is built for the bodies given in Table 1. As in the case of VSOP 82, it is represented with heliocentric elliptic variables a, λ, k, h, q, p. It is reckoned to inertial and dynamical ecliptic and equinox (γ_{Dyn}) J2000.0 defined in (1).

VSOP 87 contains the newtonian perturbations of the eight planets between themselves, the perturbations of the Moon on the Earth-Moon barycenter and on all the planets and the relativistic perturbations expressed in isotropic coordinates. For each element and each planet all the perturbations are added up and brought together in single expression.

When dealing with the planets Mercury, Venus, EMB and Mars, the perturbations of the variables k, h, q, p, of VSOP 82 have been improved for the terms of high degree with respect to time, by the polynomials taken out of the general theory by Laskar (1986).

Besides, the first order perturbations due to the Moon have been computed to a better precision. Moreover, we have computed (Bretagnon, 1984) the second order perturbations with respect to the masses by the Moon on the Earth-Moon barycenter and on all the planets. For this computation, we have used the entire solution ELP2000-82 (Chapront-Touzé and Chapront, 1983).

All these modifications on VSOP 82 mainly improve the validity time-span of the solutions for Mercury, Venus, EMB and Mars. For these planets, the validity time-span is brought from 1000 to 4000 yr before and after J2000.0. In the ends of the time-span, the precision is about 1″.

For Jupiter and Saturn, we have ensured a precision always better than 1″ over 2000 yr before and after J2000.0. For Uranus and Neptune, the same precision is ensured over 6000 years before and after J2000.0.

Over time-spans of about some centuries only around J2000.0, the precision of VSOP 87, for the whole of the bodies, is about the same as that of VSOP 82. We give the precision of the longitude over the time-span 1 900–2 100 in Table 1.

4. VSOP 87 deriving solutions

All the solutions we are going to present in this paragraph have been built from VSOP 87. They have the same precision as this solution.

4.1. VSOP 87A solution

This solution is built for the bodies given in Table 1. It is represented with heliocentric rectangular variables X, Y, Z. It is reckoned to inertial and dynamical ecliptic and equinox J2000.0 defined in (1).

We must first solve Kepler's equation in order to get the expressions of the variables X, Y, Z:

$$\mathcal{E} - k\sin\mathcal{E} + h\cos\mathcal{E} = \lambda$$

where the longitude $\mathcal{E}$ is defined by:

$$\mathcal{E} = E + \varpi$$

E being the eccentric anomaly. We proceed through iteration as in what has been explained in Bretagnon (1981).

We afterwards get the expression of the true longitude w under the following form:

$$r\cos w = -ak + a(1-h^2\psi)\cos\mathcal{E} + ahk\psi\sin\mathcal{E}$$

$$r\sin w = -ah + ahk\psi\cos\mathcal{E} + a(1-k^2\psi)\sin\mathcal{E}$$

where r is the radius vector and:

$$\psi = \frac{1}{1+\sqrt{1-h^2-k^2}}$$

Finally, the rectangular coordinates are obtained with the following expressions:

$$X = (1-2p^2)r\cos w + 2pqr\sin w$$

$$Y = (1-2q^2)r\sin w + 2pqr\cos w$$

$$Z = -2\sqrt{1-p^2-q^2}\,(pr\cos w - qr\sin w)$$

If we call S and Sun, E the Earth, M the Moon and B the Earth-Moon barycenter, the vector Sun-Earth is given by:

$$\mathbf{SE} = \mathbf{SB} + \mathbf{BE}$$

with

$$\mathbf{BE} = \frac{\mu}{1+\mu}\mathbf{EM}$$

where μ is the ratio of the mass of the Moon over the mass of the Earth:

$$\mu = \frac{m_M}{m_E} = 0.012\,300\,02.$$

The vector $\mathbf{EM}$ has been built from the solution ELP2000-82.

As an example, we give hereunder an excerpt of the series corresponding to the heliocentric coordinates X, Y, Z of the Earth reckoned to the dynamical ecliptic and equinox J2000.0. In the series, we have only retained the terms over 10^{-6} au. This level of truncation ensures a precision of 2″ over the time-span 1900–2100. X, Y, Z are expressed in au and given according to the second form of the formula (2):

$$\begin{aligned}
X = {} & 0.005\,6114 + 0.001\,234\,T \\
& + 0.999\,829\,3\cos(1.753\,485\,7 + 6\,283.075\,850\,T) \\
& + T\times 0.000\,011\cos(2.02 + 6\,283.1\,T) \\
& + \sum_{i=1}^{38} A_i\cos(B_i + C_i T) + \sum_{i=39}^{40} T\times A_i\cos(B_i + C_i T)
\end{aligned}$$

$$\begin{aligned}
Y = {} & -0.024\,427\,0 + 0.000\,930\,T \\
& + 0.999\,892\,1\cos(0.182\,658\,9 + 6\,283.075\,850\,T) \\
& + 0.000\,005\,5\cos(3.96 + 5\,507.6\,T) \\
& + 0.000\,001\,2\cos(5.45 + 9437.8\,T) \\
& + T\times 0.000\,005\cos(5.83 + 6\,283.1\,T) \\
& + \sum_{i=1}^{38} A_i\cos\left(B_i - \frac{\pi}{2} + C_i T\right) \\
& + \sum_{i=39}^{40} T\times A_i\cos\left(B_i - \frac{\pi}{2} + C_i T\right)
\end{aligned}$$

$$\begin{aligned}
Z = {} & 0.000\,054\,T \\
& + 0.000\,002\,8\cos(3.20 + 84\,334.7\,T) \\
& + 0.000\,001\,0\cos(5.42 + 5\,507.6\,T) \\
& + T\times 0.002\,278\cos(3.413\,7 + 6\,283.076\,T) \\
& + T\times 0.000\,019\cos(3.37 + 12\,566.2\,T)
\end{aligned}$$

The quantities A_i, B_i, C_i pour $i=1, 2, \ldots, 40$ are given in Table 3.

4.2. *VSOP 87B solution*

This solution is built for the bodies given in Table 1. It is represented with heliocentric spherical variables longitude, latitude and radius vector. It is reckoned to inertial and dynamical ecliptic and equinox J2000.0 defined in (1).

The longitude L, the latitude B and the radius vector r are built through successive approximations from the rectangular variables of VSOP 87A.

We have:

$$r^2=X^2+Y^2+Z^2$$

and we define ρ by:

$$\rho^2=X^2+Y^2$$

We need to express under the form of Fourier series and Poisson series the quantities $1/\rho$ and $1/r$. We can write:

$$r^2=X^2+Y^2+Z^2=r_0^2+\delta(r^2)$$

where r_0 is a constant and $\delta(r^2)$ a series ranging about the eccentricity, and therefore small with respect to r_0^2. $1/r$ is then obtained by the development of the following expression:

$$\frac{1}{r}=\frac{1}{r_0}\left[1+\frac{\delta(r^2)}{r_0^2}\right]^{-\frac{1}{2}}$$

which converges without difficulty. In the same way we obtain the development of $1/\rho$ in Fourier series and Poisson series.

a) Computation of the longitude L:

We compute L by iteration. At the iteration $m+1$, we have:

$$L_{m+1}=L_m+\delta L_{m+1};$$

δL_{m+1} being small, we can write:

$$\delta L_{m+1}\simeq\sin\delta L_{m+1}=\frac{1}{\rho}(Y\cos L_m-X\sin L_m).$$

We take as first approximation:

$$L_0=\lambda^0+NT$$

The process converges without any difficulty. We stop the computation when δL_m is smaller than the requested precision.

b) Computation of the latitude B:

We proceed in the same way as for the longitude:

$$\rho=r\cos B;\quad Z=r\sin B$$

From an approximation B_m of B, we have:

$$\delta B_{m+1}=\frac{1}{r}(Z\cos B_m-\rho\sin B_m)$$

and

$$B_{m+1}=B_m+\delta B_{m+1}.$$

We take as first approximation:

$$B_0=0$$

We immediately get:

$$B_1=\delta B_1=\frac{Z}{r}$$

The process converges with no problem. We stop the computation when δB_m is smaller than the requested precision.

c) Computation of the radius vector r:

We have already established the analytical expression of r^2 and of $1/r$. We determine by product of series:

$$r=r^2\times\frac{1}{r}$$

4.3. *VSOP 87C solution*

This solution is built for the bodies given in Table 1. It is represented with heliocentric rectangular variables X, Y, Z. It is reckoned to mean ecliptic and equinox of date.

VSOP 87C solution is built from VSOP 87A which we put through a rotation function of the time representing the precession. The mean motions of the ecliptic and equator are completely defined by the quantities q and p that describe the motion of the ecliptic with respect to the ecliptic J2000.0, and by the precession in longitude p_A and by the obliquity ε_A.

We have used the expressions q, p, p_A, ε_A expanded in polynomials of the time up to degree 10 by Laskar (1986). They ensure a precision of about 1″ over 10000 years before and after J2000.0 in so far as the precession constants p_A^0 and ε_A^0 are perfectly known.

The values of these constants recommended by IAU are:

$$p_A^0=50\,290.966'' \text{ per thousands of Julian years}$$

$$\varepsilon_A^0=23°26'21.448''$$

In order to perform the rotation, function of the time, we used these expressions keeping only polynomials of the time of degree 6. The terms thus neglected are smaller than 1″ over 5000 yr before and after J2000.0. Finally we used the following expressions:

$$\begin{aligned}
q\times10^{10}&=-11\,346\,900.2\,T+123\,726.74\,T^2+12\,654.170\,T^3\\
&\quad-137.1808\,T^4-3.203\,34\,T^5+0.005\,072\,T^6\\
p\times10^{10}&=1\,018\,039.1\,T+470\,204.39\,T^2-5417.367\,T^3\\
&\quad-250.7948\,T^4+4.634\,86\,T^5+0.056\,431\,T^6\\
p_A&=p_A^0\,T+111.1971''\,T^2+0.077\,32''\,T^3-0.235\,316''\,T^4\\
&\quad-1\,805.5''\times10^{-6}\,T^5+174.51''\times10^{-6}\,T^6\\
\varepsilon_A&=\varepsilon_A^0-468.093''\,T-0.0155''\,T^2+1.999\,25''\,T^3\\
&\quad-5\,138''\times10^{-6}\,T^4-2\,496.7''\times10^{-6}\,T^5\\
&\quad-39.05''\times10^{-6}\,T^6
\end{aligned}$$

where T is reckoned in thousands of Julian years from J2000.0.

All these quantities slowly vary as functions of the time except the linear term of p_A. We thus note:

$$p_A=\xi+\gamma$$

with

$$\xi=50\,290.966''\,T=0.243\,817\,483\,530\,T,$$

γ being a function slowly varying with the time. In the VSOP 87 solutions the argument ξ is represented by a combination of the longitude of the Earth and the arguments of the Moon:

$$\xi=-\lambda_3-\lambda_9+\lambda_{12}+\pi$$

as we can check with the values in Table 2.

From the quantities q and p we easily determine the variations of the inclination i and of the node Ω of the mean ecliptic with respect to the fixed ecliptic J2000.0:

$$i \times 10^{10} = 22\,784\,955.37\,T - 162\,427.97\,T^2 - 5\,998.737\,T^3 + 13.211\,6\,T^4 - 0.540\,00\,T^5 + 0.246\,508\,T^6$$

$$\Omega = 3.052\,112\,654\,975 + 10^{10}(-420\,786\,043.17\,T + 743\,945.31\,T^2 + 275.036\,T^3 - 1\,813.065\,9\,T^4 - 34.838\,82\,T^5)$$

Let us note X, Y, Z the coordinates referring to the ecliptic and equinox J2000.0 (VSOP 87A solution) and X_D, Y_D, Z_D the coordinates referring to the ecliptic and equinox of date. We have:

$$\begin{pmatrix} X_D \\ Y_D \\ Z_D \end{pmatrix} = (A) \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

The rotation matrix (A) is obtained by the combination of the three rotations:

1) a rotation of the axes around 0_z of angle Ω
2) a rotation of the axes around 0_x of angle i
3) a rotation of the axes around 0_z of angle $-(\Omega + p_A)$

For $i = 1, 2$, the elements a_{ij} of the matrix (A) are put under the form:

$$a_{ij} = s_{ij} \sin \xi + c_{ij} \cos \xi, \quad i = 1, 2.$$

The quantities s_{ij}, c_{ij} and a_{3j} are polynomials of the time restricted to the degree 6. Taking the symmetries of the rotation matrix (A) into account, we have the following relations:

$$s_{2j} = c_{1j}; \quad c_{2j} = -s_{1j} \quad \text{for} \quad j = 1, 2, 3.$$

We give in Table 4 the 9 polynomials s_{1j}, c_{1j}, a_{3j} for $j = 1, 2, 3$. Upon applying to the VSOP 87A solution, the rotation defined by the matrix (A), we get a new version of VSOP 87 expressed in heliocentric rectangular variables reckoned to the mean ecliptic and equinox of date.

We limit the Poisson series thus obtained to the degree 5 so as to get the VSOP 87C version. This transformation ensures a precision of 0.000 17″ over the time-span 1 000–3 000 and a precision better than 1″ over 4 000 yr before and after J2000.0.

4.4. *VSOP 87D solution*

This solution is built for the bodies given in Table 1. It is represented with heliocentric spherical variables longitude, latitude and radius vector. It is reckoned to mean ecliptic and equinox of date. VSOP 87D is built from rectangular variables X, Y, Z, of VSOP 87C through the same transformation as what enabled us to go from VSOP 87A to VSOP 87B.

Expressions representing the motion of the Sun and of the planets Mercury, Venus and Mars over the time span $-4\,000$, $+8\,000$ (Bretagnon and Simon, 1986) were indeed built from VSOP 87D.

4.5. *VSOP 87E solution*

This solution is built for the bodies given in Table 1. It is represented with rectangular variables X, Y, Z. It is reckoned to the barycenter of the solar system and to inertial and dynamical ecliptic and equinox J2000.0. The coordinates of the Sun given with respect to the barycenter have been determined using the values of the IAU masses. We remind these values giving in Table 5 the ratio of the mass of the Sun over the mass of the planet.

The coordinates $\dot{X}$, $\dot{Y}$, $\dot{Z}$ of the velocity of the Earth with respect to the barycenter of the solar system, which are necessary to the calculation of the aberration, are easily obtained from formulas similar to (3). The results concur with the series established at the precision of 5×10^{-8} au/day by Ron and Vondrak (1986).

Table 3. Periodic terms of the heliocentric variables X and Y of the Earth given in the ecliptic and equinox J2000.0

i	$A_i (\times 10^7)$	B_i	C_i
1	83527	1.710 33	12 566.1517
2	1047	1.667 2	18 849.228
3	311	0.669	83 996.85
4	256	0.586	529.69
5	214	5.191	−1 577.34
6	171	0.495	6 279.55
7	171	6.153	6 286.60
8	144	3.472	2 352.87
9	111	2.586	−5 223.69
10	93	6.07	12 036.5
11	90	3.18	10 213.3
12	74	4.37	398.1
13	68	2.22	4 705.7
14	66	1.31	5 753.4
15	61	5.38	6 812.8
16	57	2.15	1 059.4
17	55	1.46	14 143.5
18	54	0.79	775.5
19	51	4.44	7 860.4
20	45	6.09	5 884.9
21	45	1.28	6 256.8
22	45	5.37	6 309.4
23	41	0.54	6 681.2
24	26	2.27	12 168.0
25	26	1.45	709.9
26	23	1.24	7 058.6
27	23	3.01	−4 694.0
28	22	4.51	11 506.8
29	21	5.85	11 790.6
30	20	4.07	17 789.8
31	18	2.97	796.3
32	18	6.24	6 283.1
33	18	0.40	6 283.0
34	16	1.62	25 132.3
35	16	1.42	5 486.8
36	15	0.87	213.3
37	13	5.22	7 079.4
38	13	4.80	3 738.8
39	5150	6.003	12 566.15
40	129	5.96	18 849.2

5. Number of terms of the solutions

The numbers of terms of the periodic series and the Poisson series for the various versions of VSOP 87 depend on the truncation

Table 4. Coefficients of the polynomials of the matrix (A). (Unit: 10^{-12})

Degree	0	1	2	3	4	5	6
s_{11}	0	0	−538 867 722	−270 670	1 138 205	8 604	−813
c_{11}	10^{12}	0	−20 728	−19 147	−149 390	−34	617
s_{12}	-10^{12}	0	2 575 043	−56 157	140 001	383	−613
c_{12}	0	0	−539 329 786	−479 046	1 144 883	8 884	−830
s_{13}	0	2 269 380 040	−24 745 348	−2 422 542	78 247	−468	−134
c_{13}	0	−203 607 820	−94 040 878	2 307 025	37 729	−4 862	25
a_{31}	0	203 607 820	94 040 878	−1 083 606	−50 218	929	11
a_{32}	0	2 269 380 040	−24 745 348	−2 532 307	27 473	643	−1
a_{33}	10^{12}	0	−2 595 771	37 009	1 236	−13	0

Table 5. Mass of the Sun over mass of the planet

Planet	Mercury	Venus	EMB	Mars	Jupiter	Saturn	Uranus	Neptune
Mass^{-1}	6 023 600	408 523.5	328 900.5	3 098 710	1 047.355	3 498.5	22 869	19 314

Table 6. Number of periodic terms higher than 10^{-9} in the VSOP 87 solutions

	Mercury	Venus	Earth	Mars	Jupiter	Saturn	Uranus	Neptune
VSOP 87	1163	1529	2111	3441	3297	6763	8987	4994
VSOP 87A	1426	1044	1457	2773	2308	3730	3146	1651
VSOP 87B	1299	813	1129	2376	1836	3145	3139	1307
VSOP 87C	1713	1245	1724	3258	2760	4514	4087	1775
VSOP 87D	1304	803	1119	2330	1754	2857	2354	1202
VSOP 87E	2296	1745	2063	3176	2376	3732	3075	1539

precision. These precisions, counted in relative value, deal with the quantity $T^{\alpha}(|S|+|K|)$ of the formula (2).

The truncation is 0.5×10^{-10} for Mercury, 5×10^{-10} for Venus, the Earth, Mars and the Earth-Moon barycenter, 10^{-9} for Jupiter, Saturn and Neptune, 1.6×10^{-9} for Uranus. For the outer planets, the truncation of the Poisson series corresponds to a time-span of 1 000 years ($T=1$), for the other planets a time-span of 2 000 years ($T=2$).

In VSOP 87E, the barycentric coordinates of the Sun are truncated at 2×10^{-11} au.

In order to compare the respective size of the various representations of VSOP 87 for each body, we give, in Table 6, the number of periodic terms higher than 10^{-9} in relative value (1.6×10^{-9} for Uranus).

For Mercury, for example, we find there are 1 163 periodic terms higher than 10^{-9} (0.387×10^{-9} au for the semi-major axis) for all 6 variables a, λ, k, h, q, p of the VSOP 87 solution.

Still for Mercury, there are 1 426 terms higher than 0.387×10^{-9} au for the 3 heliocentric variables X, Y, Z reckoned to ecliptic J2000.0.

We can note that, apart from Mercury, the rectangular variables solutions (VSOP 87A) contain far less terms than the elliptic variables ones (VSOP 87), particularly for the outer planets. The VSOP 87B solution expressed in longitude, latitude, radius vector even contains less terms. Finally we can note that the barycentric solution (VSOP 87E) contains many more terms than the heliocentric one (VSOP 87A) and even more, since the planet is close to the Sun.

The entire solutions withhold, of course, more terms than what is shown in Table 6 because they also contain the Poisson series and because the truncation precision is smaller than 10^{-9} for the inner planets.

6. Conclusion

The different versions are available on magnetic tape. They are accompanied with Fortran subroutine substituting time in the series and computing the derivatives with respect to time.

The series are organized according to $|S|+|K|$ decreasing and thus we can take out from them secondary series that give a weaker precision analytical representation.

If n is the number of retained terms and A the amplitude of the smallest retained term, the accuracy of the thus truncated series is about $\eta\sqrt{n}\times A$ where η is a number smaller than 2.

Acknowledgements. We deeply thank J.-L. Simon for the fruitful discussions we had over the course of this work. We also thank

Dr. E.M. Standish for his suggestions, and especially for what regards the relations situating the dynamical equinox of our solutions with respect to the equinox of DE200. The calculations have been made on the computers of the *Centre Inter-Régional de Calcul Electronique* du CNRS (F-91405 Orsay).

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