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New Computational Methods for Solving Problems of the Astronomical Vessel Position

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In this paper, a simplified and direct computation method formulated by the fixed coordinate system and relative meridian concept in conjunction with vector algebra is developed to deal with the classical problems of celestial navigation. It is found that the proposed approach, the Simultaneous Equal-altitude Equation Method (SEEM), can directly calculate the Astronomical Vessel Position (AVP) without an additional graphical procedure. The SEEM is not only simpler than the matrix method but is also more straightforward than the Spherical Triangle Method (STM). Due to tedious computation procedures existing in the commonly used methods for determining the AVP, a set of optimal computation procedures for the STM is also suggested. In addition, aimed at drawbacks of the intercept method, an improved approach with a new computation procedure is also presented to plot the celestial line of position without the intercept. The improved approach with iteration scheme is used to solve the AVP and validate the SEEM successfully. Methods of solving AVP problems are also discussed in detail. Finally, a benchmark example is included to demonstrate these proposed methods.

KEY WORDS

1. Astronomical vessel position. 2. Spherical trigonometry. 3. Intercept method.

1. INTRODUCTION. One of the important issues that arise for navigators in every voyage is the daily determination of the astronomical vessel position (AVP). However, computational approaches are not the dominant methods used either in current maritime education or in practical operation [1]. The key point for solving the AVP is to determine the vessel position by observing the celestial bodies such as the sun, moon, planets and stars. The intercept method is commonly adopted to find the AVP and is only available for the condition of the altitude of a single celestial body. This method adopts the assumed position (AP) to solve the celestial line of position (LOP) and then to determine the AVP. Following the intercept method, numerous ingenious solutions to the problem of celestial navigation have been devised and available solutions using the concept of the

AP to this problem are essentially trial-and-error methods [2,3]. These methods, whether directly or indirectly in their calculating procedures, are inevitably driven into the graphical procedure for the LOP [3,4]. Another candidate to solve the AVP problem is the Spherical Triangle Method (STM), which can calculate the AVP directly without an additional graphical procedure [5,6]. However, in all the literature different formulae are adopted for the steps of the computation procedures making it necessary to seek an optimal procedure. Moreover, although the STM is a direct computation approach, the solving process is still indirect.

To date, several packages based on numerical schemes, such as STELLA by the U.S. Navy [3] and ASTROLAB in conjunction with ALMICANTART by the French Naval Academy [7], have been developed and most of the direct computation methods can be categorized into STM and matrix. Although the latter is a direct method owing to its mathematical formulation, a four-order equation is formulated by using plane analytic geometry [2], and this method leads to complicated mathematical operations in the solving process. Since the result is nearly impossible to calculate by means of a calculator, it needs to be implemented by the numerical program for the AVP. Accordingly, it is necessary to develop a direct and simpler computation method for AVP problems.

This paper is organized as follows. Comments on the respective methodologies are given in Section 2. Section 3 describes a set of optimal computation procedures for STM. Derivations of SEEM and its further applications are included in Section 4. Section 5 presents computation procedures of the improved approach and the SEEM. Section 6 offers a benchmark example to demonstrate and validate the proposed methods. A summary with some concluding remarks is in Section 7.

2. COMMENTS ON CONVENTIONAL METHODS FOR AVP PROBLEMS.

2.1. *Sight Reduction Method for the Condition of a Single Celestial Body.* The basic concept of the sight reduction method is attributed to the circle of equal altitude and the observation itself consists of measuring the altitude of a celestial body and noting the time. In general, there exist two approaches for sight reduction: the high-altitude observation and the intercept method. For their solving processes, the former is a type of direct graphical method, while the latter is a calculation method with graphic procedures. For the high-altitude observation the requirement is to plot the celestial circle of position (COP), and the plotting elements are the geographical position (GP) of the celestial body and its co-altitude. The intercept method, in contrast, is to plot the celestial LOP and the plotting elements are the AP, computed azimuth of the celestial body, Zn, and the intercept, a. Two flowcharts for solving the COP and LOP by using the high-altitude observation and the intercept method, respectively, can be found in [8].

2.2. *Spherical Triangle Method for the Condition of Two Celestial Bodies.* Two centres of circles of equal altitude are given as S_1 and S_2, and their radii are zd_1 and zd_2, respectively. Assuming two intersections of the two circles of equal altitude be P_1 and P_2, respectively, and one of them is the AVP, then the elevated pole, P_{nr}, S_1 and S_2 can form three arcs of great circles on one another, as shown in Figure 1. Thus, there are five known parameters to solve the AVP: the two zenith distances of two celestial

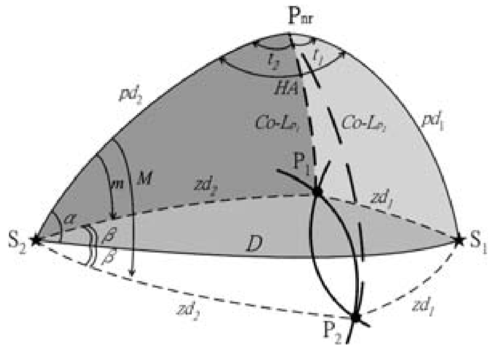


Figure 1. Obtaining an AVP using STM.

bodies (zd_1 and zd_2), the two pole distances of two celestial bodies (pd_1 and pd_2), and the difference of hour angle between two celestial bodies (HA). The solving procedures using the STM are summarized: (All the symbols are listed in the appendix.)

- *Step 1*. For the spherical triangle $\widehat{\Delta}P_{nr}S_2S_1$, if pd_1, pd_2 and HA are known, then solve D.
- *Step 2*. For the spherical triangle $\widehat{\Delta}P_{nr}S_2S_1$, if pd_1, pd_2 and HA or pd_1, pd_2 and D are known, then solve the angle α.
- *Step 3*. For the spherical triangle $\widehat{\Delta}P_1S_2S_1$ or $\widehat{\Delta}P_2S_2S_1$, if zd_1, zd_2 and D are known, then solve the angle β.
- *Step 4*. For the vessel position P_1 in the spherical triangle $\widehat{\Delta}P_{nr}S_2P_1$, the difference of angles m is equal to $(\alpha \sim \beta)$; while for the vessel position P_2 in the spherical triangle $\widehat{\Delta}P_{nr}S_2P_2$, the sum of angles M is equal to $(\alpha + \beta)$.
- *Step 5*. For the spherical triangle $\widehat{\Delta}P_{nr}S_2P_1$ or $\widehat{\Delta}P_{nr}S_2P_2$, if pd_2, zd_2 and m (or M) are known, then solve L_{P_1} and L_{P_2}.
- *Step 6*. For the spherical angle $\widehat{\Delta}P_{nr}S_2P_1$ or $\widehat{\Delta}P_{nr}S_2P_2$, if pd_2, zd_2 and m (or M) as well as pd_2, zd_2 and L_{P_1} (or L_{P_2}) are known, then solve t_2. Finally, the longitudes of AVPs, λ_{P_1} and λ_{P_2}, can be obtained by the conversion of the meridian angle and the GHA of the celestial body S_2.

After clarifying the solving procedures of the STM, it becomes easier to understand those published papers on STMs. For instance, one can find that in Chiesa's work [5], spherical triangle equations such as the side cosine formula, four-part formula, half angle formula, sine formula and Napier's analogies, are adopted in *steps 1, 2, 3, 5, and 6*, respectively. As for the work of Kotlaric [6], transformation of the Haversine function, $Hav\ x = \sin^2 \frac{x}{2}$, is adopted in *steps 1* and *5* and the half angle formula is adopted in *steps 2, 3*, and *6*. Other related papers can also be found [9–13]. Further analysis show that except for *step 4*, any one of the spherical triangle formulae used in the above steps shows the relationship of three sides and one angle. Therefore, without consideration of error propagation existing in the steps, the side cosine formula can be adopted in every step of the procedure.

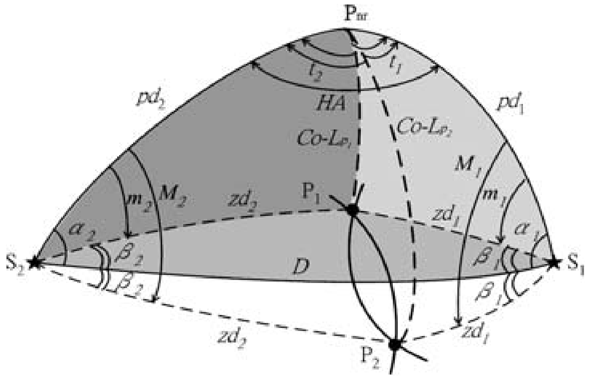


Figure 2. Obtaining an AVP using the optimal solving and checking procedures.

3. OPTIMAL COMPUTATION PROCEDURES FOR STM TO DETERMINE AVP.

3.1. *The Optimal Solving and Checking Procedures.* The STM only offers one "way" from one of the two celestial bodies to determine the AVP and the exact AVP needs an artificial judgment from a set of possible AVPs especially for potential users. If another "way" can be implemented under the same procedure, such as starting from the counterpart celestial body, a different set of parallactic (position) angles with respect to the counterpart celestial body can be possible and this results in an another set of possible AVPs before an artificial judgment. The true AVP, however, must exist in both ways simultaneously and the spurious AVPs can then be filtered out automatically without any artificial judgments if another procedure can be constructed. Based on this idea, we name this additional procedure as the checking procedure and will detail its usage in the following section.

3.1.1. *Optimal solving procedure.* From the viewpoint of celestial body S_2, the optimal solving procedure, which is to determine an AVP by using the STM, is summarized in the following (See Figure 2).

- Step 1: For the spherical triangle $\overset{\frown}{\Delta} P_{nr}S_2S_1$, D can be solved using side cosine formula,

$$\cos D = \sin d_1 \cdot \sin d_2 + \cos d_1 \cdot \cos d_2 \cdot \cos(HA). \tag{1}$$

- Step 2: Finding the angle (α_2) by using four-part formula in the spherical triangle $\overset{\frown}{\Delta} P_{nr}S_2S_1$.

$$\tan \alpha_2 = \frac{\sin(HA)}{\cos d_2 \cdot \tan d_1 - \sin d_2 \cdot \cos(HA)}. \tag{2}$$

- Step 3: Finding the angle (β_2) by using the transform type of side cosine formula in the spherical triangle $\overset{\frown}{\Delta} PS_2S_1$, i.e., $\overset{\frown}{\Delta} P_1S_2S_1$ or $\overset{\frown}{\Delta} P_2S_2S_1$.

$$\cos \beta_2 = \frac{\sin H_1 - \sin H_2 \cdot \cos D}{\cos H_2 \cdot \sin D}. \tag{3}$$

- Step 4: Finding m_2 and M_2 of the celestial body S_2.

$$m_2 = \alpha_2 \sim \beta_2;\ M_2 = \alpha_2 + \beta_2. \tag{4}$$

- Step 5: Finding L_{p_1} and L_{p_2} by using side cosine formula for the spherical triangles $\overset{\frown}{\Delta} P_{nr}S_2P_1$ and $\overset{\frown}{\Delta} P_{nr}S_2P_2$, respectively.

$$\sin L_{P_1} = \sin d_2 \cdot \cos H_2 + \cos d_2 \cdot \cos H_2 \cdot \cos m_2, \tag{5}$$

$$\sin L_{P_2} = \sin d_2 \cdot \cos H_2 + \cos d_2 \cdot \cos H_2 \cdot \cos M_2. \tag{6}$$

- Step 6: Finding t_2 by using four-part formula for the spherical triangles $\overset{\frown}{\Delta} P_{nr}S_2P_1$ and $\overset{\frown}{\Delta} P_{nr}S_2P_2$, respectively. Finally, λ_{P_1} and λ_{P_2} can be obtained by converting the meridian angle and the GHA of the celestial body S_2.

$$\tan t_2 = \frac{\sin m_2}{\cos d_2 \cdot \tan H_2 - \sin d_2 \cdot \cos m_2}, \tag{7}$$

$$\tan t_2 = \frac{\sin M_2}{\cos d_2 \cdot \tan H_2 - \sin d_2 \cdot \cos M_2}. \tag{8}$$

3.1.2. *Checking procedure.* From the viewpoint of celestial body S_1, the checking procedure, which is to determine an AVP by using STM, is summarized in the following (Refer to Figure 2).

- Step 1: For the spherical triangle $\overset{\frown}{\Delta} P_{nr}S_2S_1$, D can be solved using side cosine formula. Since this step is the same as step 1 of the optimal solving procedure, Equation (1) is used.
- Step 2: Finding α_1 by using four-part formula in the spherical triangle $\overset{\frown}{\Delta} P_{nr}S_2S_1$.

$$\tan\alpha_1 = \frac{\sin(HA)}{\cos d_1 \cdot \tan d_2 - \sin d_1 \cdot \cos(HA)}. \tag{9}$$

- Step 3: Finding β_1 by using the transform type of side cosine formula in the spherical triangle $\overset{\frown}{\Delta} PS_2S_1$, i.e., $\overset{\frown}{\Delta} P_1S_2S_1$ or $\overset{\frown}{\Delta} P_2S_2S_1$.

$$\cos\beta_1 = \frac{\sin H_2 - \sin H_1 \cdot cosD}{\cos H_1 \cdot \sin D}. \tag{10}$$

- Step 4: Finding m_1 and M_1 of the celestial body S_1, which being the difference and sum of the results of Steps 2 and 3, in the spherical triangles $\overset{\frown}{\Delta} P_{nr}S_1P_1$ and $\overset{\frown}{\Delta} P_{nr}S_1P_2$, respectively.

$$m_1 = \alpha_1 \sim \beta_1;\ M_1 = \alpha_1 + \beta_1. \tag{11}$$

- Step 5: Finding L_{p_1} and L_{p_2} by using side cosine formula for the spherical triangles $\overset{\frown}{\Delta} P_{nr}S_1P_1$ and $\overset{\frown}{\Delta} P_{nr}S_1P_2$ respectively.

$$\sin L_{P_1} = \sin d_1 \cdot \cos H_1 + \cos d_1 \cdot \cos H_1 \cdot \cos m_1, \tag{12}$$

$$\sin L_{P_2} = \sin d_1 \cdot \cos H_1 + \cos d_1 \cdot \cos H_1 \cdot \cos M_1. \tag{13}$$

- Step 6: Finding t_1 of the celestial body S_1 by using four-part formula for the spherical triangle $\overset{\frown}{\Delta} P_{nr}S_1P_1$ and $\overset{\frown}{\Delta} P_{nr}S_1P_2$ respectively. Finally, λ_{P_1} and λ_{P_2} can be obtained by converting the meridian angle and the GHA of the celestial body S_1.

$$\tan t_1 = \frac{\sin m_1}{\cos d_1 \cdot \tan H_1 - \sin d_1 \cdot \cos m_1}, \tag{14}$$

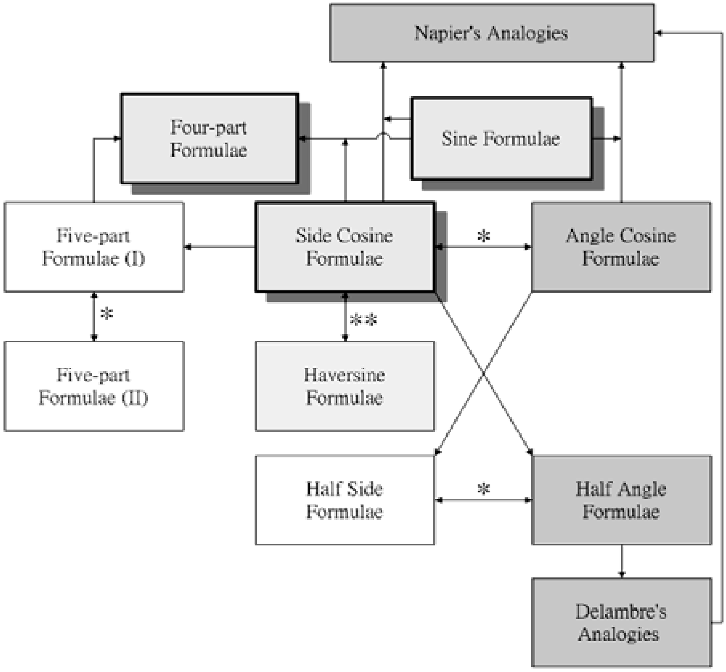
$$\tan t_1 = \frac{\sin M_1}{\cos d_1 \cdot \tan H_1 - \sin d_1 \cdot \cos M_1}. \tag{15}$$

3.2. *Discussions on Optimal Formulae.* The solving and checking procedures adopt different formulae but have the same steps; however, only the solving procedure will be discussed in this section. Also, since Step 4 deals with plus and minus operations only, it will be excluded from the discussion. Although other literatures [5,6,9–13] found useful formulae in AVP solving procedures, most of them are complex and tedious compared to our suggested optimal formulae. Despite the differences, all solving procedures originate from the fundamental formulae of spherical trigonometry with respect to their own adaptive conditions. Based on this characteristic, we discuss the optimal formulae with criteria of simplicity and minimum error propagation.

3.2.1. *Formulae of spherical trigonometry.* Side cosine formulae, sine formulae and four-part formulae are fundamental formulae in spherical trigonometry since many formulae in spherical trigonometry are derived from them. Side cosine formulae describe the relation between the three sides and any one of the angles. Therefore, if two sides and their included angle are given, the third side can be obtained from the primitive side cosine formula. In contrast, if three sides are given, any angle can be obtained by transposing the primitive formula. Sine formulae give the relation between two angles and the two sides opposite them. Since "the greater angle is opposite of the greater side, and conversely", it results in ambiguity and artificial judgments are needed when applying the formulae. As for the four-part formulae, they describe the relation of any adjacent four parts of the spherical triangle, such as the relation of any two sides with their outer angle and inner angle or that of any two angles with their outer side and inner side. Adaptive conditions of the four-part formulae are: given two sides and the included angle to find any outer angle and given two angles and the included side to find any outer side [8,14].

From a mathematical perspective, the important half angle formulae, Napier's and Delambre's analogies are dominant in spherical trigonometry due to their alternative symmetry characteristic. For instance, the half angle formulae are considered universal; however, their formulations are more complex than the side cosine formulae or the four-part formulae. While from the navigational perspective, Haversine is the commonly used formulae, derived from the side cosine formulae for logarithmic work. The above mentioned formulae are more complex than the fundamental formulae, side cosine formulae and four-part formulae. The relationships of the various formulae of spherical trigonometry are summarized and shown in Figure 3. This explains why differences in proposed formulae solving procedures exist in literatures.

3.2.2. *Choice of the optimal formulae.* The objectives for *Steps 1* and *2* are to transfer the problem into describing the navigational triangle, such that an oblique spherical triangle with two sides and the included angle are known. It is necessary



Note: * *Polar duality theorem;* ** *Equivalent.*

Figure 3. Relationships of various formulae in spherical trigonometry.

to determine the values of the third side (*Step 1*) and the outer angle (*Step 2*) respectively. In *Step 1*, the side cosine formulae are chosen according to the criterion of simplicity; while in *Step 2*, the four-part formulae are chosen according to the criterion of minimum error propagation in the solving procedures. Objectives for *Steps 5* and *6* are the same as those for *Steps 1* and *2*, therefore, *Step 5* uses the side cosine formulae and *Step 6* uses the four-part formulae. As for *Step 3*, the objective is to transfer the problem into describing navigational triangle, *i.e.*, an oblique spherical triangle with its three sides known, and to find any one of the angles. The transposes of side cosine formulae are suggested according to the criterion of simplicity. Thus, Equations (1), (3), (5), (6), (10), (12) and (13) in *Steps 1, 3* and *5* adopt the side cosine formulae and Equations (2), (7), (8), (9), (14) and (15) in *Steps 2* and *6* adopt the four-part formulae.

4. DERIVATIONS OF GOVERNING EQUATIONS FOR SEEM. First of all, since the celestial equator coordinate system is the extension of the Earth coordinate system, the celestial sphere can be considered as a unit sphere. Therefore, from the navigator's perspective, the Earth coordinate system includes the position variables, the latitude and the longitude; the celestial equator coordinate system includes the position variables, declination and GHA. These systems can replace the conventional (mathematical) spherical coordinate system. By doing so, the position vector for any point P in the Cartesian coordinate system can be

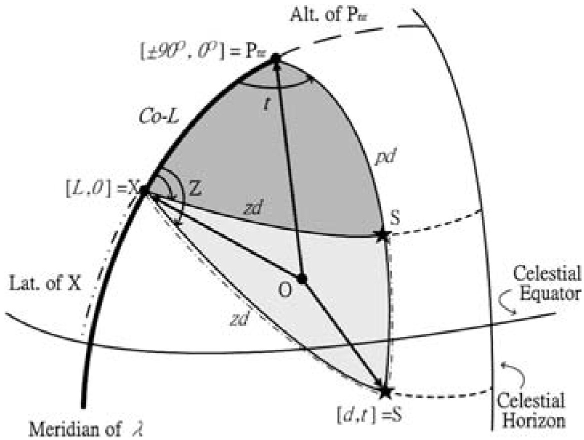


Figure 4. An astronomical triangle in the combined celestial equator and celestial horizon systems of coordinates.

expressed as

$$\begin{aligned}\vec{P} &= (\cos L \cdot \cos \lambda,\ \cos L \cdot \sin \lambda,\ \sin L) \\ &= (\cos d \cdot \cos G,\ \cos d \cdot \sin G,\ \sin d).\end{aligned} \tag{16}$$

Because the coordinate system has been decided, the sign convention of the latitude or the declination is a positive value for the north and negative for the south. The concept of the relative meridian is then introduced and the Greenwich meridian is replaced as the local meridian for transformation of the coordinate system, as shown in Figure 4. Therefore,

$$\vec{X} = (\cos L, 0,\ \sin L), \tag{17a}$$

$$\vec{S} = (\cos d \cdot \cos t,\ \cos d \cdot \sin t,\ \sin d), \tag{17b}$$

$$\vec{P}_{nr} = (0, 0, \pm 1). \tag{17c}$$

The meridian angle, t, is determined by the differences between the observer's longitude and the celestial body's GHA. The sign convention is decided according to the conventional practice of the celestial navigation. Besides, the latitude of the observer is equal to the altitude of the elevated pole and the celestial equator coordinate system and the celestial horizontal coordinate system can be combined together. In other words, the described position variables of altitude and azimuth angle of the celestial horizontal coordinate system based on the observer can be set up on the celestial sphere. Hence, the astronomical triangle can be formed. Figure 4 illustrates the three vertices, three sides and three angles.

4.1. *Equal Altitude Equation of Celestial Body.* In Figure 4, the included angle, side of astronomical triangle, of the unit vectors, $\vec{X}$ and $\vec{S}$, is exactly the zenith distance, zd, which is also called the co-altitude since the celestial body can only be observed upon the celestial horizontal plane. According to the geometric and algebraic definitions of the dot product of two vectors, one has

$$\begin{aligned}\vec{X} \bullet \vec{S} &= 1 \cdot 1 \cdot \cos(zd) = \sin H \\ &= \cos L \cdot \cos d \cdot \cos t + \sin L \cdot \sin d.\end{aligned} \tag{18}$$

Rearranging the above equation one has

$$\sin H = \sin L \cdot \sin d + \cos L \cdot \cos d \cdot \cos t. \tag{19}$$

Equation (19) is the well-known side cosine formula in spherical trigonometry. Since several formulae, such as the Haversine formula, are derived from this equation, it has been recognized as the basic formula in celestial navigation.

4.2. *Azimuth Angle Equation of Celestial Body*. Different given conditions can lead to different azimuth angle equations of the celestial body. The altitude azimuth equation, the time and altitude azimuth equation and the time azimuth equation, are derived, respectively, in the following.

4.2.1. *The altitude azimuth equation*. As shown in Figure 4, the side cosine formula in the spherical trigonometry can be expressed as

$$\cos(pd) = \sin L \cdot \cos(zd) + \cos L \cdot \sin(zd) \cdot \cos Z. \tag{20}$$

Because $pd = 90° \mp d$ and $zd = 90° - H$, putting them into Equation (20) yields

$$\sin d = \sin L \cdot \sin H + \cos L \cdot \cos H \cdot \cos Z. \tag{21}$$

Therefore, the azimuth angle can be obtained as

$$\cos Z = \frac{\sin d - \sin L \cdot \sin H}{\cos L \cdot \cos H}. \tag{22}$$

Equation (22) is another form of the side cosine formula in spherical trigonometry. When the altitude of the celestial body is given, the azimuth angle can be determined from the equation, which is thus called the altitude azimuth equation in celestial navigation.

4.2.2. *The time and altitude azimuth equation*. As shown in Figure 4, the azimuth angle is the included angle, angle of astronomical triangle, of the two unit vectors, $(\vec{X} \times \vec{S})$ and $(\vec{X} \times \vec{P}_{nr})$. The geometric and algebraic meanings of the cross products of the two vectors can be expressed as

$$\begin{aligned}||\vec{X} \times \vec{S}| \times |\vec{X} \times \vec{P}_{nr}|| &= |[\sin(zd) \cdot \sin(Co - L) \cdot \sin(Z)]\vec{X}| \\ &= \cos H \cdot \cos L \cdot \sin Z \\ &= |[(\vec{X} \times \vec{S}) \bullet \vec{P}_{nr}]\vec{X}| = \cos L \cdot \cos d \cdot \sin t.\end{aligned} \tag{23}$$

Rearranging it yields

$$\cos H \cdot \sin Z = \cos d \cdot \sin t. \tag{24}$$

Equation (24) is the sine formula in spherical trigonometry and is also called the time and altitude azimuth equation in celestial navigation. Also, a set of Equations (19) and (24) is called the sin-cosine equations or the classic equations.

4.2.3. *The time azimuth equation.* Substituting Equations (19) and (24) into Equation (21) yields

$$\sin d = (\sin d \cdot \sin L + \cos d \cdot \cos L \cdot \cos t) \cdot \sin L + \left(\frac{\cos d \cdot \sin t}{\sin Z}\right) \cdot \cos L \cdot \cos Z. \quad (25)$$

Rearranging the above equation can yield

$$\sin d \cdot (1 - \sin^2 L) = (\cos d \cdot \cos L \cdot \cos t \cdot \sin L) + \cos d \cdot \sin t \cdot \cos L \cdot \cot Z. \quad (26)$$

By dividing $\cos d \cdot \cos L$ at two sides of the above equation simultaneously yields

$$\tan d \cdot \cos L = \cos t \cdot \sin L + \sin t \cdot \cot Z. \quad (27)$$

Hence,

$$\tan Z = \frac{\sin t}{(\cos L \cdot \tan d) - (\sin L \cdot \cos t)}. \quad (28)$$

Equation (28) is the four-part formula of spherical trigonometry. When the altitude is unknown, this equation can be used to obtain the azimuth to adjust the compass error for celestial body observation. Therefore, this equation is also called the time azimuth equation in celestial navigation.

5. CONSTRUCTING NEW COMPUTATION PROCEDURES.

The computation procedures for the altitude of a single celestial body and the altitudes of two celestial bodies can be constructed by choosing different combinations of formulae to determine AVP effectively at different sight conditions.

5.1. *Condition of the Altitude of a Single Celestial Body.*

5.1.1. *Combined computation formulae of the intercept method.* The intercept method is to choose an AP near to the most possible position (MPP) and take it as the reference position to compute the altitude and the azimuth, respectively. The appropriate measure is to choose Equations (19) and (28) and one has

$$\sin H_C = \sin L \cdot \sin d + \cos L \cdot \cos d \cdot \cos t, \quad (29)$$

$$\tan Z_C = \frac{\sin t}{(\cos L \cdot \tan d) - (\sin L \cdot \cos t)}. \quad (30)$$

5.1.2. *Computation procedure to solve the LOP without intercept.* In fact, Equations (29) and (30) are generally used to make the sight reduction tables for marine navigation, more specifically, the commonly used Pub. No. 229. The AP is originated from entering arguments of integral degrees in accordance with the inspection table. If the computation method is adopted, the choice of the initial reference positions, such as the dead reckoning (DR) position, the MPP, or the optimal estimated position (EP), can be unconstrained. The distance between the AP and the true vessel position should not exceed 30 nautical miles which is the result of entering arguments of integrated degrees. Again, if the computation method is used, this impractical regulation can be released and the accuracy of the obtained AVP can be increased by the iteration method. From a computation perspective, once the computed azimuth is obtained, the perpendicular intersection can be calculated by using the classic equations. Then, the computed azimuth line and the LOP can be plotted by considering the intersection as the reference point of the possible AVP. In other words, this scheme is a kind of computation method that can plot the LOP without intercept. Hence, it is an improved approach for conventional intercept method, and the basic idea is illustrated in the following.

Equations (19) and (24) are chosen for plotting the LOP without intercept and they can further be expressed as

$$\sin t = \frac{\cos H_0 \cdot \sin Z_C}{\cos d}, \tag{31}$$

$$\sin H_0 = \sin L \cdot \sin d + \cos L \cdot \cos d \cdot \cos t, \tag{32}$$

in which Z_C can be obtained by using Equation (30). By using Equation (31), the meridian angle can be obtained quickly and a comparison of the longitude of the initial reference position and the GP can easily yield the longitude of the possible AVP. Similarly, the latitude of the possible AVP can be obtained by using Equation (32). Derivations of the trigonometric equation for the computation are presented in the following.

Dividing sin d at two sides of Equation (32) simultaneously can yield

$$\frac{\sin H_0}{\sin d} = \sin L + \frac{\cos L \cdot \cos t}{\tan d}. \tag{33}$$

Now, let

$$\tan \theta = \frac{\cos t}{\tan d}. \tag{34}$$

Substitute Equation (34) into Equation (33), multiply cosθ at both sides simultaneously and introduce the additional formula can yield

$$\sin (L + \theta) = \frac{\sin H_0 \cdot \cos \theta}{\sin d}. \tag{35}$$

Therefore, a new computation procedures to solve the LOP without intercept can be summarized in the following:

- *Step 1*. Z_C can be obtained by using Equation (30).
- *Step 2*. The longitude of the possible AVP can be obtained by using Equation (31).
- *Step 3*. The latitude of the possible AVP can be obtained by using Equations (34) and (35).
- *Step 4*. By taking the possible AVP as the reference point and plotting the azimuth line according to the computed azimuth, the LOP can be determined from the line that is perpendicular to the azimuth line and passes through the reference point simultaneously. This step is also a graphic drawing work.

5.2. *Condition of the Altitudes of Two Celestial Bodies.* The condition of the altitude of two celestial bodies can be categorized into two cases. One is the case of observing the altitudes of two celestial bodies simultaneously or nearly simultaneously; the other is the case of observing altitudes of the same or different celestial

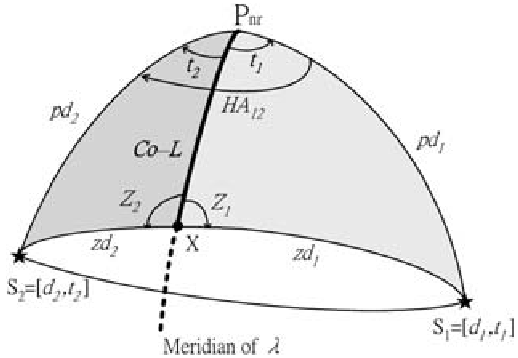


Figure 5. Obtaining an AVP using SEEM.

bodies at different time. For the latter case, the running fix concept is usually adopted and after the course, speed, and period of two observing times have been identified, the latter case can be transferred into the former one when the rhumb line sailing in conjunction with the moving reference position, the GP of high-altitude observation or the AP of intercept method, is adopted. The reason for obtaining a celestial fix is that each would have to be advanced or retired to the desired time for the fix, and making proper allowance for the travel of the ship during the intervening time [3,4]. As shown in Figure 5, since the observing altitudes and their GPs of the two celestial bodies have been determined, the equal altitude equations of the two celestial bodies can be expressed as:

$$\cos d_1 \cdot \cos t_1 \cdot \cos L + \sin d_1 \cdot \sin L = \sin H_1, \tag{36}$$

$$\cos d_2 \cdot \cos t_2 \cdot \cos L + \sin d_2 \cdot \sin L = \sin H_2. \tag{37}$$

Now let $a_1 = \cos d_1$, $b_1 = \sin d_1$, $c_1 = \sin H_1$, $a_2 = \cos d_2$, $b_2 = \sin d_2$, and $c_2 = \sin H_2$. From Equations (36) and (37), one has

$$\cos L = \frac{b_2 c_1 - b_1 c_2}{a_1 b_2 \cos t_1 - a_2 b_1 \cos t_2}, \tag{38}$$

$$\sin L = \frac{a_1 c_2 \cos t_1 - a_2 c_1 \cos t_2}{a_1 b_2 \cos t_1 - a_2 b_1 \cos t_2}. \tag{39}$$

Basically, the celestial equator coordinate system can be considered as the extension of the Earth coordinate system. Since the two systems have been integrated in the celestial sphere and are further combined with the celestial horizontal coordinate system; based on the concepts that the observer is exactly the zenith and the celestial sphere is a unit sphere, the length of the unit vector is equal to 1 by geometric definition, and the observed altitude of the zenith should be 90 degrees in celestial navigation, that means

$$\cos^2 L + \sin^2 L = 1. \tag{40}$$

Substituting Equations (38) and (39) in Equation (40) and rearranging, the relation of the meridian angles of the two celestial bodies in the combined coordinate system can be yielded as

$$(A \cdot \cos t_1 - B \cdot \cos t_2) \cdot (C \cdot \cos t_1 - D \cdot \cos t_2) = E^2, \tag{41}$$

in which

$$\left.\begin{aligned} A = a_1(b_2 - c_2) &= \cos d_1 \cdot (\sin d_2 - \sin H_2) \\ B = a_2(b_1 - c_1) &= \cos d_2 \cdot (\sin d_1 - \sin H_1) \\ C = a_1(b_2 + c_2) &= \cos d_1 \cdot (\sin d_2 + \sin H_2) \\ D = a_2(b_1 + c_1) &= \cos d_2 \cdot (\sin d_1 + \sin H_1) \\ E = b_2 c_1 - b_1 c_2 &= \sin d_2 \cdot \sin H_1 - \sin d_1 \cdot \sin H_2 \end{aligned}\right\}. \tag{42}$$

Since now the celestial bodies can be observed simultaneously, the difference of meridian angles for the two celestial bodies can be expressed as (refer to Figure 5)

$$HA_{12} = t_1 \sim t_2. \tag{43}$$

Taking the cosine operation in Equation (43) yields

$$\cos t_1 = p \cdot \cos t_2 + q \cdot \sin t_2, \tag{44}$$

in which

$$\left.\begin{aligned} p &= \cos HA_{12} \\ q &= \sin HA_{12} \end{aligned}\right\}. \tag{45}$$

Substituting Equation (44) into Equation (41) yields

$$[(Ap - B) \cdot \cos t_2 + Aq \cdot \sin t_2] \cdot [(Cp - D) \cdot \cos t_2 + Cq \cdot \sin t_2] = E^2. \tag{46}$$

To solve t_2, from Equation (46), let

$$\tan\alpha = \frac{Ap - B}{Aq}, \tag{47}$$

in which

$$\left.\begin{aligned} \sin\alpha &= \frac{Ap - B}{R} \\ \cos\alpha &= \frac{Aq}{R} \\ R &= \pm\sqrt{(Ap - B)^2 + (Aq)^2} \end{aligned}\right\}, \tag{48}$$

and also let

$$\tan\beta = \frac{Cp - D}{Cq}, \tag{49}$$

in which

$$\left.\begin{aligned} \sin\beta &= \frac{Cp - D}{S} \\ \cos\beta &= \frac{Cq}{S} \\ S &= \pm\sqrt{(Cp - D)^2 + (Cq)^2} \end{aligned}\right\}. \tag{50}$$

The sign convention of R is the same as that of the multiply product of $Ap-B$ and Aq. Similarly, the sign convention of S is determined by that of the multiply product of $Cp-D$ and Cq. Substituting Equations (47) and (49) into Equation (46) and introducing the additional formulae, one has

$$\sin(t_2+\alpha)\cdot\sin(t_2+\beta)=\frac{E^2}{RS}. \tag{51}$$

Introducing the products of trigonometric functions in Equation (51) yields

$$\cos(2t_2+\alpha+\beta)=\cos(\alpha-\beta)-\frac{2E^2}{RS}. \tag{52}$$

Now, t_2 can be obtained and reduced to the longitude of the AVP. By using Equation (43), t_1, can be obtained. Therefore, the computation procedure of the SEEM can be summarized in the following.

- *Step 1*. The preliminaries, A, B, C, D, E, p, q, R and S, can be obtained by using Equations (42), (45), (48), and (50).
- *Step 2*. The parameters, α and β, can be obtained from Equations (47) and (49).
- *Step 3*. The t_2, can be obtained from Equation (52) and further reduced to the longitude of the AVP, λ. Also, t_1, can be determined from Equation (43).
- *Step 4*. Repeating uses of Equations (34) and (35) with respect to Equations (36) and (37), respectively, can determine the latitudes of the AVPs for celestial body S_1 and celestial body S_2, respectively. Also, the results can be checked with each other for validation.

6. AN ILLUSTRATIVE EXAMPLE AND DISCUSSIONS ON SOLVING AVP PROBLEMS.

6.1. *Illustrative Example. The 2004 DR position of a vessel is $L41°34{\cdot}8'N$, $\lambda017°00{\cdot}5'W$. At 20-03-58, the star* ***Capella*** *is observed with a sextant. At 20-02-56, shortly before the above observation, another star, the* ***Alkaid*** *is spotted. The navigator records the needed information and further reduces it from the nautical almanac for sight reduction as shown in Table 1.*

6.2. *Available Methods.* The AVP can be determined by using the following four approaches for sight reduction.

1. Using the intercept method together with the inspection table to solve the *AP*, *Zn* and *a*, and plot the LOP. (Approach 1)
2. Treating DR as the initial reference position, using the new computation method without intercept to solve the possible AVP and plot the LOP. (Approach 2)
3. Using STM to determine the AVP directly.
4. Using SEEM to determine the AVP directly.

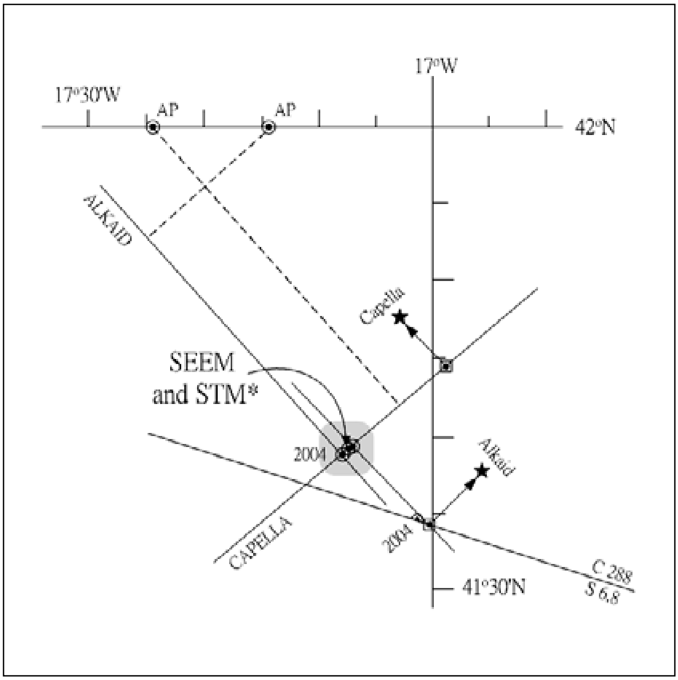
6.3. *Approach 1.* The three elements, *AP, Zn* and *a*, for plotting the LOPs from the inspection table and needed information are summarized in Table 2. They can be shown in the small-area plotting sheet to plot the LOPs and further determine the graphical AVP, $L41°38{\cdot}6'N$, $\lambda017°08{\cdot}1'W$, as shown in Figure 6.

Table 1. Extract of relevant information from [4] *Dutton's Navigation and Piloting*, pp. 384–385 (1985).

Body	ZT	Ho	GP
Capella	20-03-58	15°19·3′	45°58·4′N 131°24·8′W
Alkaid	20-02-56	77°34·9′	49°25·7′N 003°14·2′W

Table 2. Three elements, AP, Zn and a, for plotting the LOPs by using intercept method (approach 1).

Body	The three plotting elements of LOP
Capella	$AP\begin{pmatrix} 42°N & Zn=318{\cdot}8° \\ 017°24{\cdot}8'W & a=24{\cdot}2'Away \end{pmatrix}$
Alkaid	$AP\begin{pmatrix} 42°N & Zn=047{\cdot}9° \\ 017°14{\cdot}2'W & a=10{\cdot}4'Away \end{pmatrix}$



Note: * *Spherical Triangle Method (STM).*

Figure 6. A comparison of the various methods for sight reduction.

6.4. *Approach 2.* Use of the proposed improved approach without intercept can obtain the computed azimuth and possible AVP as the reference point. The procedures and results are summarized in Table 3 and they can be used to determine the graphical AVP, $L41°39{\cdot}4'N$, $\lambda 017°06{\cdot}9'W$, as shown in Figure 6.

Table 3. The solution of the improved method (approach 2) for obtaining LOPs.

Process \ Item	Eq.	Input	Output	Solution
(Capella) 1	(30)	$t=114°24{\cdot}3'W(Est.)$ $L=41°34{\cdot}8'(DRL)$ $d=45°58{\cdot}4'(GPL)$	$Zc=N40{\cdot}98587018°W$	$Zn=319°$
2	(31)	$Ho=15°19{\cdot}3'$ $Zc=40{\cdot}98587018°$ $d=45°58{\cdot}4'$	$t=65{\cdot}52879657°$ *or* $114{\cdot}47120343°W$*	$\lambda=016°56{\cdot}5'W$
3	(34)	$t=114°28{\cdot}3'$ $d=45°58{\cdot}4'$	$\theta=-21{\cdot}82130093°$	
	(35)	Ho, θ, d	$L+\theta=19{\cdot}94797852°$	$L=41°46{\cdot}2'N$
(Alkaid) 1	(30)	$t=13°46{\cdot}3'E(Est.)$ $L=41°34{\cdot}8'(DRL)$ $d=49°25{\cdot}7'(GPL)$	$Zc=N46{\cdot}10682304°E$	$Zn=46{\cdot}1°$
2	(31)	$Ho=77°34{\cdot}9'$ $Zc=46{\cdot}10682304°$ $d=49°25{\cdot}7'$	$t=13{\cdot}78447652°E$	$\lambda=017°00{\cdot}3'W$
3	(34)	$t=13°47{\cdot}1'E$ $d=49°25{\cdot}7'$	$\theta=39{\cdot}74635188°$	
	(35)	Ho, θ, d	$L+\theta=81{\cdot}32013587°$	$L=41°34{\cdot}4'N$
Answer	Capella	$\begin{pmatrix}41°46{\cdot}2'N \\ 016°56{\cdot}5'W\end{pmatrix}$, $Zn=319°$		
	Alkaid	$\begin{pmatrix}41°34{\cdot}4'N \\ 017°00{\cdot}3'W\end{pmatrix}$, $Zn=46{\cdot}1°$		

* Since $\sin\theta=\sin(180°-\theta)$, either θ or $(180°-\theta)$ can be chosen according to the estimated t.

6.5. *STM Method.* Use of the STM can directly determine the AVP, $L41°39{\cdot}1'N$, $\lambda017°07{\cdot}3'W$, without plotting. Results and the procedures are listed in Table 4 and the computed AVP is shown in Figure 6.

6.6. *SEEM Method.* Use of SEEM can directly determine the AVP, $L41°39{\cdot}1'N$, $\lambda017°07{\cdot}3'W$, without plotting. Results and the procedures are listed in Table 5 and the computed AVP is shown in Figure 6.

6.7. *Comparison of results.* As can be seen in Figure 6, both the intercept method (approach 1) and the improved method (approach 2) with the graphic procedures can determine the AVP. Besides, the STM and the SEEM can also be used to solve the AVP successfully as shown in Tables 4 and 5, respectively. Since plotting the LOP is unnecessary for both of the latter two approaches, they are more direct than and superior to the former ones. Moreover, results from this example also validate theories of the proposed two computation approaches.

6.8. *Validation.* Figure 7 shows the computed results of the four approaches along the LOP of the star Capella in a larger scale than Figure 6, it can be seen that the computed AVPs are not nearly the same. Therefore, the improved approach (approach 2) with the iteration method is proposed to validate the true AVP. The details of computing results are listed in Table 6. As can be seen in Figure 7, when using the AVPs obtained from approaches 1 and 2, respectively as the initial values for iterations, both approach quickly to the AVP obtained by using the SEEM and STM. It proves that the SEEM and STM are more accurate than

Table 4. The solution using STM to obtaining an AVP.

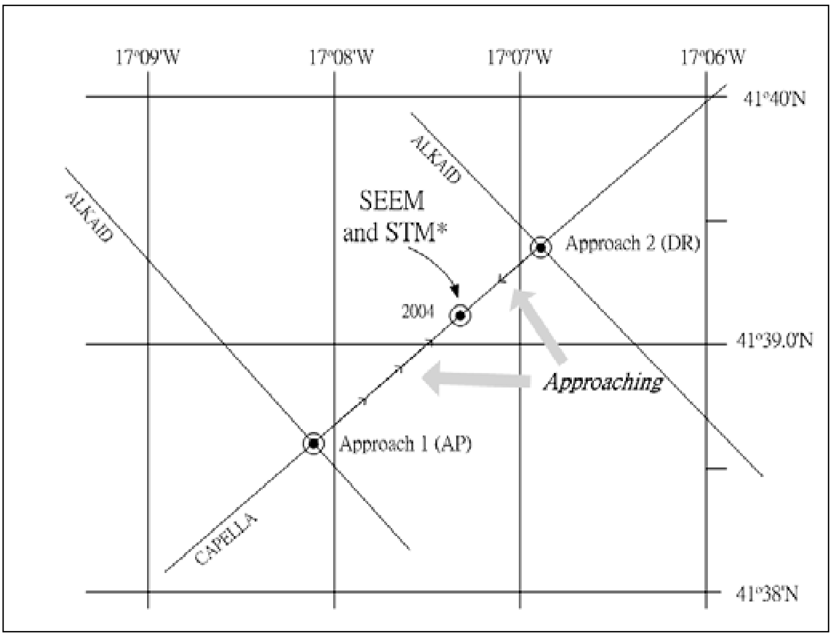
Process \ Item	Eq.	Input	Output	Solution
Capella-2				
1	(1)	$d_1=49°25{\cdot}7'$ $d_2=45°58{\cdot}4'$	$D=74{\cdot}52784803°$	
2	(2)	$HA=128°10{\cdot}6'W$	$\alpha_2=32{\cdot}03989161°$	
3	(3)	$H_1=77°34{\cdot}9'$ $H_2=15°19{\cdot}3'$ $D=74{\cdot}52784803°$	$\beta_2=12{\cdot}88160258°$	
4	(4)	$m_2=\alpha_2\sim\beta_2$ $M_2=\alpha_2+\beta_2$	$m_2=19{\cdot}15828903°$ $M_2=44{\cdot}92149419°$	
5	(5)	d_2, H_2, m_2	$L_{P_1}=55{\cdot}40227535°$	$L_{P_1}=55°24{\cdot}1'N$
	(6)	d_2, H_2, M_2	$L_{P_2}=41{\cdot}65224667°$	$L_{P_2}=41°39{\cdot}1'N$
6	(7)	d_2, H_2, m_2	$t_2=-33{\cdot}87823373°$ $=146{\cdot}12176627°W$ *	$\lambda_{P_1}=041°42{\cdot}5'E$
	(8)	d_2, H_2, M_2	$t_2=-65{\cdot}70854453°$ $=114{\cdot}29145547°\ W$ *	$\lambda_{P_2}=017°07{\cdot}3'W$
Answer		$P_1=\begin{pmatrix}55°24{\cdot}1'N\\014°42{\cdot}5'E\end{pmatrix};\ P_2=\begin{pmatrix}41°39{\cdot}1'N\\017°07{\cdot}3'W\end{pmatrix}$		
Alkaid-1				
1	(1)	$d_1=49°25{\cdot}7'$ $d_2=45°58{\cdot}4'$	$D=74{\cdot}52784803°$	
2	(9)	$HA=128°10{\cdot}6'W$	$\alpha_1=34{\cdot}53321078°$	
3	(10)	$H_1=77°34{\cdot}9'$ $H_2=15°19{\cdot}3'$ $D=74{\cdot}52784803°$	$\beta_1=88{\cdot}97451004°$	
4	(11)	$m_1=\alpha_1\sim\beta_1$ $M_1=\alpha_1+\beta_1$	$m_1=54{\cdot}44129926°$ $M_1=123{\cdot}50772082°$	
5	(12)	d_1, H_1, m_1	$L_{P_1}=55{\cdot}40227536°$	$L_{P_1}=55°24{\cdot}1'N$
	(13)	d_1, H_1, M_1	$L_{P_2}=41{\cdot}65224669°$	$L_{P_2}=41°39{\cdot}1'N$
6	(14)	d_1, H_1, m_1	$t_1=17{\cdot}9450996°E$	$\lambda_{P_1}=021°10{\cdot}9'W$
	(15)	d_1, H_1, M_1	$t_1=13{\cdot}8852112°E$	$\lambda_{P_2}=017°07{\cdot}3'W$
Answer		$P_1=\begin{pmatrix}55°24{\cdot}1'N\\021°10{\cdot}9'W\end{pmatrix};\ P_2=\begin{pmatrix}41°39{\cdot}1'N\\017°07{\cdot}3'W\end{pmatrix}$		

* Since $\tan(-\theta)=\tan(180°-\theta)$, therefore $(-\theta)$ is replaced as $(180°-\theta)$.

the other two approaches. Furthermore, they are more versatile than the other two approaches especially for higher altitude observation conditions due to their theoretical background. Moreover, the SEEM and STM can compute the AVP effectively without graphic procedures. In Figure 7, it also has been found that the practical measured distance between the AVPs from approach 1 and the SEEM and STM are over 0·5 nautical mile. This significant difference shows that curvature errors exist as a result of the replacement of the COP by the LOP at the higher altitude of the star Alkaid in the intercept method. However, when the improved approach is adopted in conjunction with the iteration method, the accurate AVP can also be obtained by the direct computation without considering the two assumptions of the intercept method as shown in Figure 7 and Table 6.

Table 5. The solution using SEEM to obtain an AVP.

Item / Process	Eq.	Input	Output	Solution
preliminary			$A=-0{\cdot}1508207907$	
			$B=0{\cdot}2957874168$	
		$d_1=45°58{\cdot}4'$	$C=1{\cdot}206644605$	
	(42)	$d_2=49°25{\cdot}7'$	$D=0{\cdot}6395072223$	
	(45)	$H_1=15°19{\cdot}3'$	$E=-0{\cdot}5014807812$	
	(48)	$H_2=77°34{\cdot}9'$	$p=-0{\cdot}618088309$	
	(50)	$HA_{12}=128°10{\cdot}6'E$	$q=0{\cdot}7861086708$	
			$R=0{\cdot}234712942$	
			$S=-1{\cdot}678947942$	
1	(47)	A, B, p, q	$\alpha=59{\cdot}65973955°$	
	(49)	C, D, p, q	$\beta=-55{\cdot}59985116°$	
2	(52)	E, R, S, and $\cos(\alpha-\beta)$ $=-0{\cdot}4267201302$	$2t_2+\alpha+\beta$ $=31{\cdot}83031073°$ $t_2=13{\cdot}88521117°$ $=13°53{\cdot}1'E$	$\lambda=017°07{\cdot}3'W$
	(43)	HA_{12}, t_2	$t_1=114°17{\cdot}5'W$	
3	(34)	t_1, d_1	$\theta=-21{\cdot}68459739°$	$L=41{\cdot}65238413°$
	(35)	H_1, θ, d_1	$L+\theta=19{\cdot}96778674°$	$\lambda=41°39{\cdot}1'N$
Check	(34)	t_2, d_2	$\theta=39{\cdot}73424525°$	$L=41{\cdot}65208024°$
	(35)	H_2, θ, d_2	$L+\theta=81{\cdot}38632549°$	$\lambda=41°39{\cdot}1'N$
Answer		Astronomical Vessel Position: $L=41°39{\cdot}1'N$, $\lambda=017°07{\cdot}3'W$		



Note: * *Spherical Triangle Method (STM).*

Figure 7. AVPs obtained the various methods in a larger scale.

Table 6. The solution of the improved method (approach 2) in conjunction with iteration.

Body / Iteration	Reference point	Capella	Alkaid	Intersection
Approach 1	AP, Z_n, a			
0		$\begin{pmatrix} 42°N & Zn=318{\cdot}8° \\ 017°24{\cdot}8'W & a=24{\cdot}2'A \end{pmatrix}$	$\begin{pmatrix} 42°N & Zn=047{\cdot}9° \\ 017°14{\cdot}2'W & a=10{\cdot}4'A \end{pmatrix}$	$\begin{pmatrix} 41°38{\cdot}6'N \\ 017°08{\cdot}1'W \end{pmatrix}$
1	$\begin{pmatrix} 41°38{\cdot}6'N \\ 017°08{\cdot}1'W \end{pmatrix}$	$\begin{pmatrix} 41°38{\cdot}6'N \\ 017°08{\cdot}1'W \end{pmatrix}, Z_n=318{\cdot}9°$	$\begin{pmatrix} 41°39{\cdot}0'N \\ 017°07{\cdot}2'W \end{pmatrix}, Z_n=46{\cdot}5°$	$\begin{pmatrix} 41°39{\cdot}2'N \\ 017°07{\cdot}4'W \end{pmatrix}$
2	$\begin{pmatrix} 41°39{\cdot}2'N \\ 017°07{\cdot}4'W \end{pmatrix}$	$\begin{pmatrix} 41°39{\cdot}0'N \\ 017°07{\cdot}5'W \end{pmatrix}, Z_n=318{\cdot}9°$	$\begin{pmatrix} 41°39{\cdot}2'N \\ 017°07{\cdot}4'W \end{pmatrix}, Z_n=46{\cdot}5°$	$\begin{pmatrix} 41°39{\cdot}2'N \\ 017°07{\cdot}3'W \end{pmatrix}$
3	$\begin{pmatrix} 41°39{\cdot}2'N \\ 017°07{\cdot}3'W \end{pmatrix}$	$\begin{pmatrix} 41°39{\cdot}1'N \\ 017°07{\cdot}3'W \end{pmatrix}, Z_n=318{\cdot}9°$	$\begin{pmatrix} 41°39{\cdot}2'N \\ 017°07{\cdot}4'W \end{pmatrix}, Z_n=46{\cdot}5°$	$\begin{pmatrix} 41°39{\cdot}1'N \\ 017°07{\cdot}3'W \end{pmatrix}$
4	$\begin{pmatrix} 41°39{\cdot}1'N \\ 017°07{\cdot}3'W \end{pmatrix}$	$\begin{pmatrix} 41°39{\cdot}1'N \\ 017°07{\cdot}3'W \end{pmatrix}, Z_n=318{\cdot}9°$	$\begin{pmatrix} 41°39{\cdot}1'N \\ 017°07{\cdot}3'W \end{pmatrix}, Z_n=46{\cdot}5°$	$\begin{pmatrix} 41°39{\cdot}1'N \\ 017°07{\cdot}3'W \end{pmatrix}$
Answer	Astronomical Vessel Position: $L=41°39{\cdot}1'N$, $\lambda=017°07{\cdot}3'W$			
Approach 2	DR			
0	$\begin{pmatrix} 41°34{\cdot}8'N \\ 017°00{\cdot}5'W \end{pmatrix}$	$\begin{pmatrix} 41°46{\cdot}2'N \\ 016°56{\cdot}5'W \end{pmatrix}, Z_n=319°$	$\begin{pmatrix} 41°34{\cdot}4'N \\ 017°00{\cdot}3'W \end{pmatrix}, Z_n=46{\cdot}1°$	$\begin{pmatrix} 41°39{\cdot}4'N \\ 017°06{\cdot}9'W \end{pmatrix}$
1	$\begin{pmatrix} 41°39{\cdot}4'N \\ 017°06{\cdot}9'W \end{pmatrix}$	$\begin{pmatrix} 41°39{\cdot}4'N \\ 017°06{\cdot}9'W \end{pmatrix}, Z_n=318{\cdot}9°$	$\begin{pmatrix} 41°39{\cdot}2'N \\ 017°07{\cdot}4'W \end{pmatrix}, Z_n=46{\cdot}5°$	$\begin{pmatrix} 41°39{\cdot}1'N \\ 017°07{\cdot}3'W \end{pmatrix}$
2	$\begin{pmatrix} 41°39{\cdot}1'N \\ 017°07{\cdot}3'W \end{pmatrix}$	$\begin{pmatrix} 41°39{\cdot}1'N \\ 017°07{\cdot}3'W \end{pmatrix}, Z_n=318{\cdot}9°.$	$\begin{pmatrix} 41°39{\cdot}1'N \\ 017°07{\cdot}3'W \end{pmatrix}, Z_n=46{\cdot}5°$	$\begin{pmatrix} 41°39{\cdot}1'N \\ 017°07{\cdot}3'W \end{pmatrix}$
Answer	Astronomical Vessel Position: $L=41°39{\cdot}1'N$, $\lambda=017°07{\cdot}3'W$			

7. CONCLUSIONS.

In this paper, the SEEM method using the fixed coordinate system and relative meridian concept with the vector algebra has been constructed to deal with the AVP problems successfully. Also, a set of optimal computation procedures for the STM has been suggested to fix the AVP problems and a new computation procedure to solve the LOP without the intercept has been proposed to replace the commonly used intercept method. Results of this new procedure with the iteration method are thus used to validate the proposed SEEM successfully. Consequently,

- Based on the criteria of simplicity and minimum error propagation, optimal formulae in spherical trigonometry are suggested to construct a method for solutions and to form the STM to resolve the AVP. In addition, a checking procedure originating from the counterpart celestial body is included in the STM to solve the true AVP without the need for artificial judgments.
- Unlike the STM of an indirect method, the SEEM of a direct one can be used to determine the AVP automatically without graphical procedures.
- To overcome the drawbacks of the intercept method, which is essentially a trial-and-error method, we have developed an improved method with a new computation procedure to plot LOP without use of the intercept. With the iteration scheme this improved method can determine the true AVP.

- Both the STM and the SEEM are more accurate than the intercept method and the improved method with a new computation procedure. In addition, both of these are also more versatile than the other two methods especially when the higher altitude observation conditions are encountered.

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APPENDIX

S_1, S_2	two celestial bodies as shown in Figures 1, 2 and 5
D	great circle distance between two celestial bodies
M, m	parallactic angle of the celestial body S_2 as shown in Figure 1
L_{P_1}	latitude of AVP (P_1)
L_{P_2}	latitude of AVP (P_2)
t_2	meridian angle of the celestial body S_2
λ_{P_1}	longitude of AVP (P_1)
λ_{P_2}	longitude of AVP (P_2)
d_1	declination of celestial body S_1
d_2	declination of celestial body S_2
α_2	angle of the spherical triangle $\widehat{\Delta P_{nr} S_2 S_1}$ as shown in Figure 2
H_1	observed altitude of celestial body S_1

H_2	observed altitude of celestial body S_2
β_2	angle of the spherical triangles $\widehat{\Delta}P_1S_2S_1$ or $\widehat{\Delta}P_2S_2S_1$ as shown in Figure 2
m_2	parallactic angle of the spherical triangle $\widehat{\Delta}P_{nr}S_2P_1$ as shown in Figure 2
M_2	parallactic angle of the spherical triangle $\widehat{\Delta}P_{nr}S_2P_2$ as shown in Figure 2
α_1	angle of the spherical triangle $\widehat{\Delta}P_{nr}S_2S_1$ as shown in Figure 2
β_1	angle of the spherical triangles $\widehat{\Delta}P_1S_2S_1$ or $\widehat{\Delta}P_2S_2S_1$ as shown in Figure 2
m_1	parallactic angle of the spherical triangle $\widehat{\Delta}P_{nr}S_1P_1$ as shown in Figure 2
M_1	parallactic angle of the spherical triangle $\widehat{\Delta}P_{nr}S_1P_2$ as shown in Figure 2
t_1	meridian angle of the celestial body S_1
L	latitude of an observer (or the AVP)
$\lambda_¡$	longitude of an observer (or the AVP)
d	declination of a celestial body
G	GHA of a celestial body
$\vec{X}$	position vectors for the zenith X as shown in Figure 4
$\vec{S}$	position vectors for the celestial body S as shown in Figure 4
$\vec{P}_{nr}$	position vectors for the elevated pole P_{nr} as shown in Figure 4
t	meridian angle
H	altitude of a celestial body
zd	zenith distance
pd	polar distance
Z	azimuth angle
H_C	computed altitude
Z_C	computed azimuth angle
H_0	observed altitude
HA_{12}	difference of meridian angles from celestial body S_1 to celestial body S_2 as shown in Figure 5